Geometry + Technology = Proof

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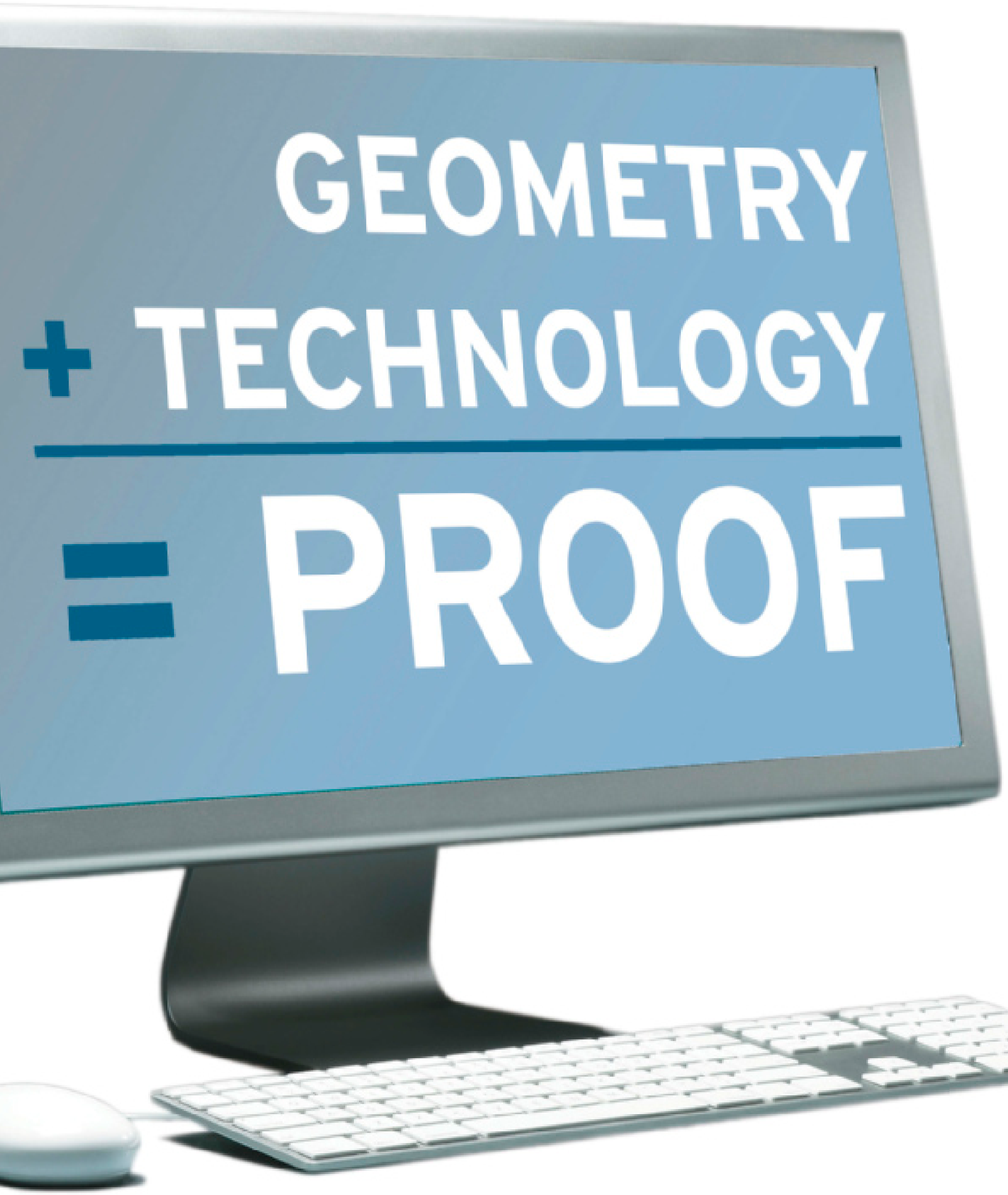

GEOMETRY
+ TECHNOLOGY
= PROOF

Symbolic geometry software, such as Geometry Expressions, can guide students as they develop strategies for proofs.

Irina Lyublinskaya and Dan Funsch

Several interactive geometry software packages are available today to secondary school teachers. An example is The Geometer's Sketchpad® (GSP), also known as Dynamic Geometry® software, developed by Key Curriculum Press. This numeric based technology has been widely adopted in the last twenty years, and a vast amount of creativity has been brought to bear on applying dynamic geometry software (DGS) to the educational process (Gawlick 2002; King and Schattschneider 1997). DGS allows students to discover results for themselves, formulate conjectures and intermediate results, examine special cases, and generate new ideas (Scher 1999; De Villiers 1999, 2006; Steckroth 2005). GeoGebra and TI-Nspire™ computer algebra systems have dynamic geometry capabilities and a built-in computer algebra system (CAS); however, CAS does not have the capability to establish algebraic relationships between geometric objects and their properties.

Geometry Expressions™ (GE), developed by Saltire Software, is the first of a new class of interactive symbolic geometry system. This software takes a geometric configuration and outputs algebraic expressions for quantities measured from the model (Todd 2007). Integrating geometric and algebraic explorations could be a powerful tool for helping students develop reasoning skills in the inductive exploration-based approach (Majewski 2007).

The Geometer's Sketchpad, GeoGebra, and TI-Nspire all allow students to explore geometric objects visually and dynamically and to generate and confirm conjectures on the basis of their observations. This is an important step in developing proofs, and the value of these software packages cannot be underestimated. We may say that these software packages provide a geometric approach to strengthening reasoning skills.

Geometry Expressions, on the other hand, has the capability to produce symbolic algebraic outputs for geometric objects, thus providing opportunities for developing an algebraic approach to proofs. We will discuss several examples of how symbolic geometry can be used to guide students as they develop strategies for proofs. The accompanying examples of student work illustrate this process.

FINDING AND COMPARING SYMBOLIC EXPRESSIONS

Consider how Geometry Expressions can help students solve the following problem:

> The altitude to the hypotenuse of a right triangle will divide it into two segments. What is the relationship between the altitude to the hypotenuse and these two segments?

In this problem, students will use the technique of finding and comparing symbolic expressions.

As students measure the lengths of each segment and the altitude, Geometry Expressions (GE) provides algebraic expressions for these quantities in terms of the legs of the triangle, rather than simply

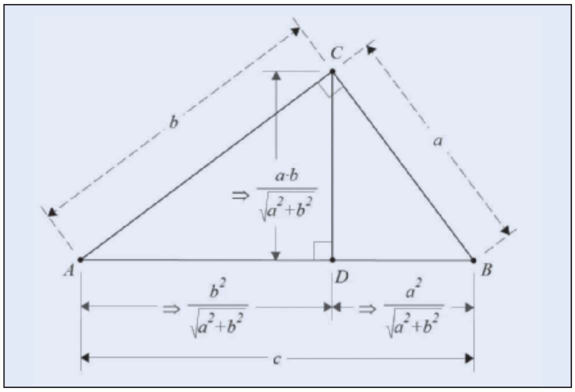
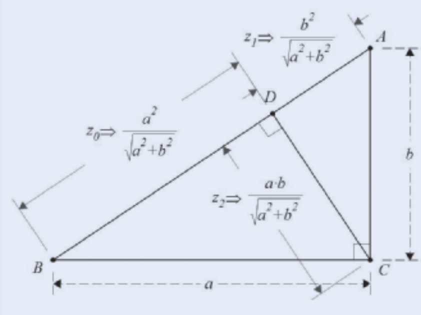


Fig. 1 Geometry Expressions uses algebra in a geometric setting.

a numeric value (see **fig. 1**). The accompanying student work (see **fig. 2**) clearly documents the way in which one high school student used GE to help her develop a strategy to prove this theorem. After constructing the figure and investigating the algebraic expressions for the lengths of several segments, the student, in her words, "realized that if [the appropriate segment lengths] were multiplied together, then the product would be the square of the altitude."

This work, of course, does not constitute a proof. But we can see how, by expressing the segment lengths as algebraic expressions, the software allowed the student to perceive a relationship. When she stated that she "needed to figure out

The first thing I noticed about these triangles was that the symbolic measurement all had the same denominator. While this proved to be a relatively useless observation in terms of finding the relationship, it did however help me solve my proof. I then focused in on the numerators of the two segments and realized that if they were multiplied together, [then] the product would be the square of the altitude. I tried out this idea on the entire measurement and it turned out to be true. When I began writing my proof, I realized that I first had to prove that the triangles were similar. Then I realized that I needed to figure out how those symbolic measurements were created. After much thought I realized that my answer could be found using the proportions between the sides of the triangles. After that it was simply crunching the numbers.



Prove: $\sqrt{(BD)(AD)} = CD$
Given: $\angle BCA$ and $\angle ADC$ both equal 90°; BC is constrained to be a and CA to b.

Proof:	**Reasons:**
1. Triangle ADC is similar to triangle BCA.	Two congruent angles
2. Triangle CDB is similar to triangle BCA.	Two congruent angles
3. $\frac{AB}{a} = \frac{a}{BD}$	Proportional sides of similar triangles
4. $BD = \frac{a^2}{\sqrt{a^2+b^2}}$	Pythagorean theorem + algebra
5. $\frac{AB}{b} = \frac{b}{AD}$	Proportional sides of similar triangles
6. $AD = \frac{b^2}{\sqrt{a^2+b^2}}$	Pythagorean theorem + algebra
7. $\frac{AB}{b} = \frac{a}{CD}$	Proportional sides of similar triangles
8. $CD = \frac{ab}{\sqrt{a+b^2}}$	Pythagorean theorem + algebra
9. $\sqrt{\frac{a^2}{\sqrt{a^2+b^2}} \frac{b^2}{\sqrt{a^2+b^2}}} = \frac{ab}{\sqrt{a^2+b^2}}$	Algebra
10. $\sqrt{(BD)(AD)} = CD$	Substitution

Fig. 2 A twelfth-grade student constructed this proof for this problem: *CD* is a geometric mean of *AD* and *BD*.

how those symbolic measurements were created," she was well on her way to discerning the contribution each parameter makes to the altitude length. Specifically, she saw that she might consider the roles played by proportions as well as by the Pythagorean theorem. It is doubtful that these elements would have been so apparent had the software provided simply the numerical values for the lengths.

Another approach to this problem is to find an expression for the area of the right triangle. When asked to find the area of the triangle, students always find it as "base times height":

$$A = \frac{1}{2}ch = \frac{h\sqrt{a^2 + b^2}}{2}$$

However, students usually have a hard time seeing that the area can also be represented as half the product of the legs of the (right) triangle. Using GE to determine the area, students will obtain the expression $A = (1/2)ab$ (see **fig. 3**). They can set both expressions for the area equal to each other and determine that

$$h = \frac{ab}{\sqrt{a^2 + b^2}}.$$

Now, students will need to justify the expressions for the lengths of segments AD and BD to complete the proof. Using the Pythagorean theorem, $AD = \sqrt{b^2 - h^2}$, and $BD = \sqrt{a^2 - h^2}$, they can now find expressions for these segments and complete the proof.

Figure 4 contains two additional examples in which the technique of finding and comparing symbolic expressions may be applied to the development of proofs. The proofs of these statements are left to the readers.

FILL IN THE BLANKS

Now consider the technique that we call "fill in the blanks." We will use this technique to illustrate how Geometry Expressions can help with proving the central angle theorem.

If we draw an angle θ at the center of a circle and another angle at the circumference subtending the same arc, GE jumps straight to an expression for the inscribed angle in terms of the central angle (see **fig. 5**). This may be the correct answer, but it does not constitute a proof. The software helps students work out a sequence of intermediate results, which, taken together, do constitute a proof.

In class, students began by considering the special case when chord DE is a diameter. In the accompanying student work (see **fig. 6**), we can see that, after querying GE for the algebraic expressions for

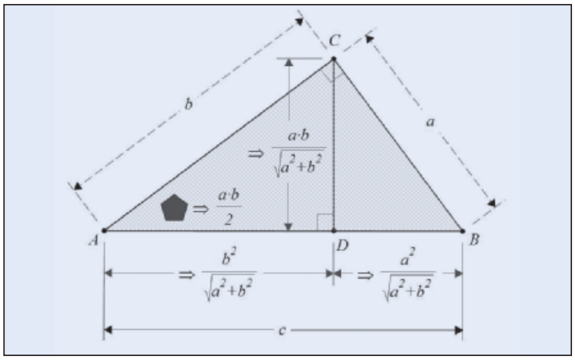


Fig. 3 An expression for the area of the triangle appears near vertex *A*.

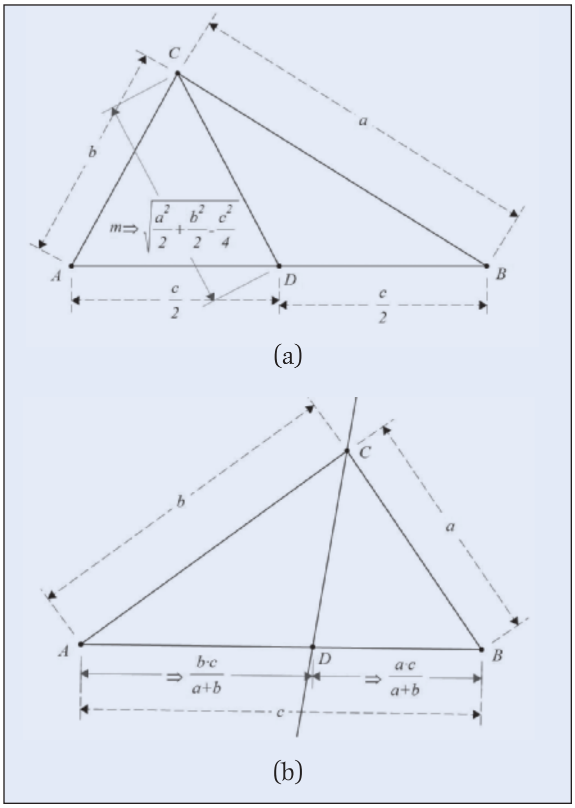


Fig. 4 Theorems about a triangle's median (**a**) and its angle bisector (**b**) are hinted at in these figures.

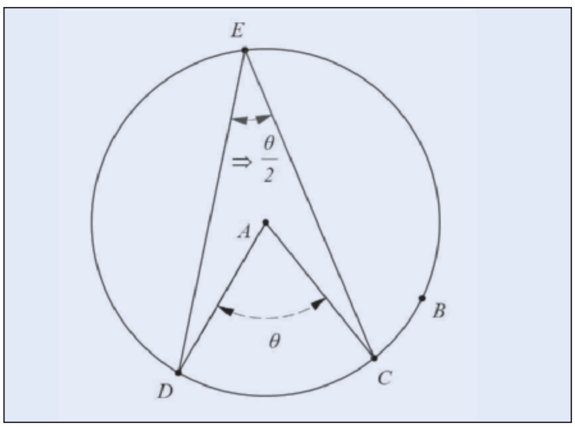


Fig. 5 Expressions for the relationship between the angles *DEC* and *DAC* do not constitute a proof.

I began by defining the central angle, CAD, as θ. I immediately observed that the adjacent supplementary angle, CAE, would be $(180 - \theta)°$, or, in the case of a circle, $\pi - \theta$. Logically I concluded that if one of the three angles in the triangle equaled $\pi - \theta$, then the other two, ACE and AEC, added together, must equal θ in order to have a total angle measurement of π. I then proceeded to determine the measurements of these remaining two angles. After observing $\triangle CAE$ for some time, I noticed that segments CA and EA, two of the sides of the triangle, had to be congruent, because they were both radii of the circle. A triangle with two congruent sides sounded very familiar. I then remembered other qualities of an isosceles triangle, particularly that the base angles are congruent. This meant that the two remaining angles I was trying to determine had the exact same measurement. If the two angle measurements added together were θ, then the measurement of each of the two angles had to be half of θ, or $\theta/2$. This proved the conjecture that if a central angle equals θ, then the inscribed angle that subtends the same arc equals $\theta/2$.

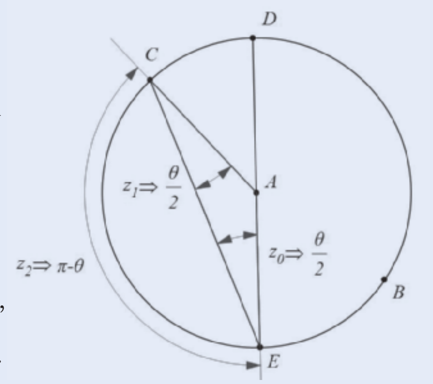


Fig. 6 A twelfth-grade student proved the special case of the inscribed angle theorem in which the side of the angle is a diameter of the circle.

the measures of the angles, one student "noticed that ... the sides of the triangle had to be congruent." This observation is related to his subsequent recollection of the theorem about the base angles of an isosceles triangle. He combines all these concepts and writes a proof. His work demonstrates that the information provided by GE guided his thoughts about the relationships between the various elements and, ultimately, helped him discover a path to his proof.

In a follow-up lesson, students considered the general case (see **fig. 7**). The key step, for both a paper-and-pencil proof and a proof using GE, was to add the diameter EF into the diagram. Having added diameter EF and specified angle DAF as α and angle CAF as β, one can ask the software to display the corresponding inscribed angles. Angle DEF is $\alpha/2$, and angle CEF is $\beta/2$, as proved earlier. Then,

$$\angle DAC = \alpha + \beta = 2\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = 2\angle DEC.$$

Figure 8 contains two further examples in which this technique may be applied. These are natural extensions of the central angle theorem, in which angles are formed by the secant lines with the vertex of an angle outside the circle (see **fig. 8a**) and by the secant lines with the vertex of an angle inside the circle (see **fig. 8b**).

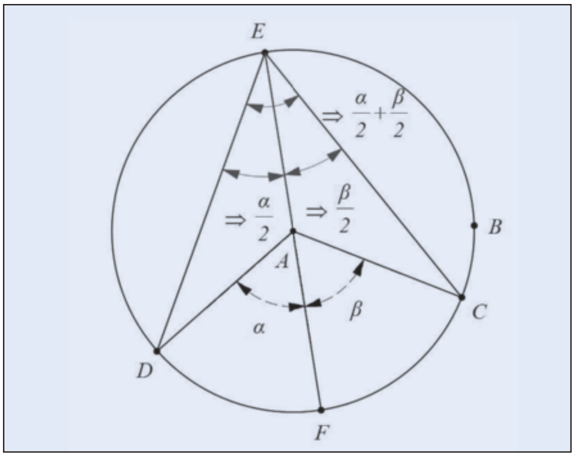


Fig. 7 Drawing the auxiliary diameter *EF* helps students prove the theorem.

MATHEMATICAL INDUCTION

In the previous examples, we used the symbolic geometry output as scaffolding for the construction of a traditional mathematical proof. Now we take a different approach. We accept the results of the symbolic geometry system as correct and usable in our proof without further checking, but we extend a set of special cases to a general result using mathematical induction. In **figure 9**, we start with two circles of radius 1 tangent to a circle of radius 2 and each other, and from this we create a sequence of circles each tangent to three given circles. Using our symbolic geometry system, we can derive the radii of these circles. They form the following sequence:

$$\frac{2}{3}, \frac{1}{3}, \frac{2}{11}, \frac{1}{9}, \frac{2}{27}, \ldots$$

A first question to ask would be, What is the next element of the sequence? We might be inspired to notice that the numerator alternates between 1 and 2. We may decide to treat each as a separate sequence, or we may choose to change the denominator so that all the numerators are 2, thus obtaining this sequence:

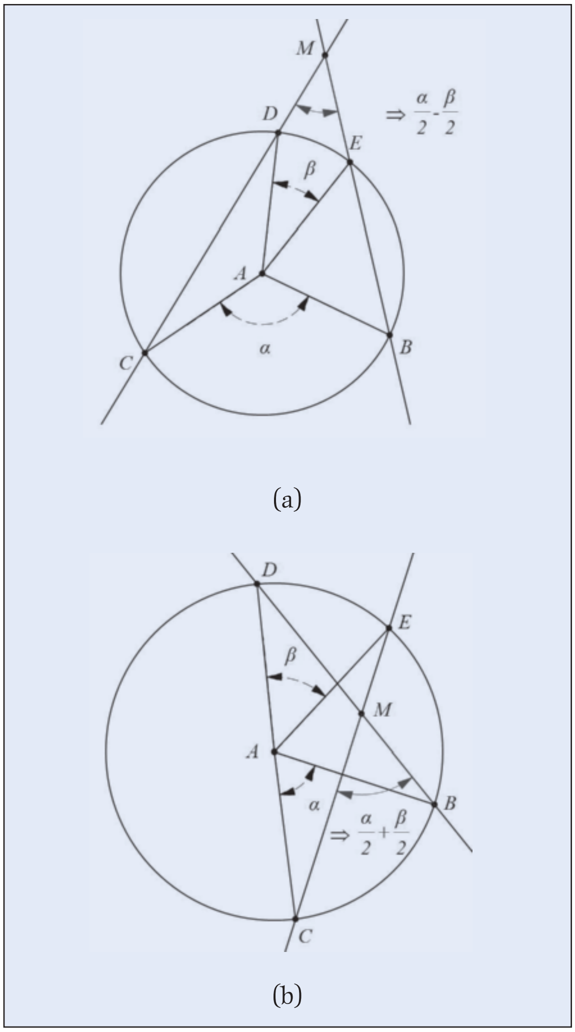


Fig. 8 The software helps students see how to measure both secant-secant angles (**a**) and chord-chord angles (**b**).

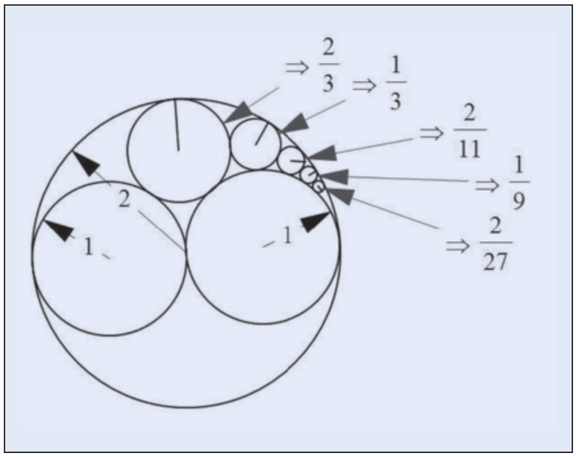


Fig. 9 Constructing enough circles enables students to see the pattern.

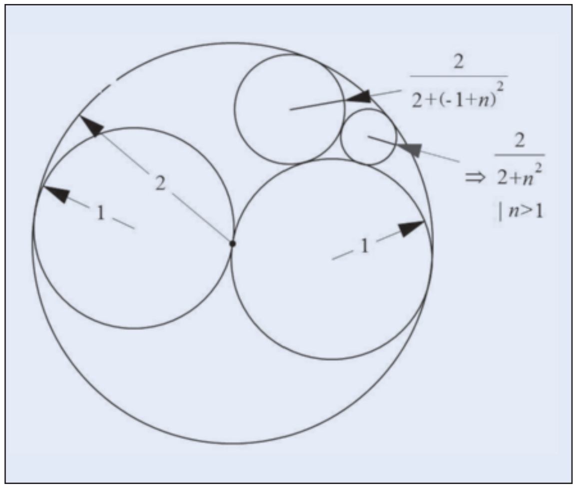


Fig. 10 The general situation for the inductive step has the $(n - 1)$th circle with radius $2/[(n - 1)^2 + 2]$ tangent to the circles with radii 2 and 1 (shown) and the $(n - 2)$th circle (not shown).

$$\frac{2}{3}, \frac{2}{6}, \frac{2}{11}, \frac{2}{18}, \frac{2}{27}, \ldots$$

With this change, perhaps, the pattern in the denominators becomes clearer. We could elicit a general form for the radius: $2/(n^2+2)$.

The question now arises of how we can use Geometry Expressions to prove this general result. Is it possible to prove the result for any value of n? To do so, we need to introduce the powerful proof technique of mathematical induction. Using this technique, we need first to prove that the theorem holds for an initial element of our sequence and then to prove that if the theorem holds for all values up to $n - 1$, it also holds for n.

The anchor is easy to show:

$$\frac{2}{1^2+2}=\frac{2}{3}$$

To prove the general step using GE, we need to create the generic situation (see **fig. 10**), in which the $(n - 1)$th element of the sequence of circles is tangent to the circle of radius 2 and one of the circles of radius 1 and has the prescribed radius

$$\frac{2}{(n-1)^2+2}.$$

We create the next circle in the sequence, which is tangent to the circle of radius 2, the circle of radius 1, and the circle of radius

$$\frac{2}{(n-1)^2+2}.$$

GE shows that this new circle has radius $2/(n^2+2)$, completing the inductive proof.

In this example, the symbolic geometry software allows the proof technique of mathematical induction to be illustrated without the need to perform significant algebraic manipulation. Accepting the truth of the result for $n - 1$ corresponds to setting the radius of the given circle to

$$\frac{2}{(n-1)^2+2}$$

while retaining two of its tangencies. Solving the general step involves the simple symbolic geometry task of specifying a circle tangent to three given circles and asking GE for its radius. Again, although the manipulation required of the student is negligible, the steps of an inductive proof are all present.

CONCLUSION

In this article, we have considered several examples of using Geometry Expressions to facilitate the process of proofs based on algebraic approach. As seen in the examples presented here, GE helps complete the proofs in the following ways:

- It provides the value or the formulas of the goal parameter.
- It provides the value and the formulas of the introduced parameter.

One strength of a symbolic geometry system in teaching proof is its use in assisting the creation of conjectures. The tool can help students break a problem into tractable parts.

GE gives an algebraic form of the relationship between various parameters, allowing students to formulate arbitrarily complicated intermediate results. GE provides the ability both to jump to a solution and to investigate the algebraic form of the relationship between geometric objects in the problem. A typical way to use this software in this context is first to state the theorem in such a way that making a symbolic measurement constitutes a computer "proof." If a mathematical proof is not apparent, we make other measurements and, if necessary, additional constructions and use these to suggest a potential proof path. Although symbolic geometry measurements can be used in formulating purely geometrical proofs, they can also be used to bridge the components of a geometric-algebraic proof.

The main emphasis of this article is on the use of symbolic geometry software to aid in creating proof (as shown in examples 1 and 2), but we have also discussed the use of the software in a computer-aided proof mode (example 3). In this latter style, the software's output is taken as an acceptable component of a proof. Proof by induction is an important and conceptually challenging topic. The ability to present it in a partially geometric context and to have the algebraic manipulation done automatically should have significant pedagogic benefits.

Our experience with Geometry Expressions in the secondary school classroom suggests that by providing a method of automatically generating intermediate results, the software can help students with the strategic planning of a proof. Thus, it can make them more independent of teacher-provided hints. Along with independence, students may gain ownership of the problem and its solution as well as the motivation to push through to a fully realized mathematical proof.

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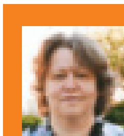

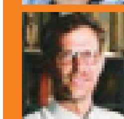

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