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The laundry-bucket nephroid, a punch to the kidney and some student-level mathematical research.

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On a relaxing Sunday morning I strode into the laundry, for some purpose or other, flicked the light on, and found a laundry bucket strategically place in my path. I could have tripped and fallen, was my first reaction. But, as I looked into the bucket I saw the most romantic thing I had seen for sometime, a naturally occurring heart shape (cardioid) in the base of the laundry bucket. I thought what a lovely thing for my wife to do. I thought she knew very little mathematics so this was a surprise. When I said thank you to her she said, "You mistaken fool, that is a nephroid, not a cardioid. A nephroid is the shape of a kidney, which is where I am going to punch you unless you get on with the weeks worth of washing in the laundry." So much for romance!

Figure 1: That laundry bucket and the nephroid or kidney-shaped curve caused by reflecting light rays.

As I walked to the living room, in the hope of a distraction that would result in a task other than the washing, I pondered the shape I had seen. I had always thought it was a cardiod. I quietly retired to the WWW and, after some reading, found my wife was sort of correct. If the light source was infinitely far away from the bucket, the shape would be a nephroid, so what we were seeing was not really a nephroid, and probably not a cardioid either, but I was not sure. I thought better of telling my wife this at the time and decided to dub it the laundry-bucket nephroid.

In the living room my two daughters had appeared with their Spirograph set. One had two wheels, one with a radius of twice the other and she had the larger fixed and rolled the smaller one around it. She said, "Look Daddy!" I looked and saw the image seen in Figure 2. I asked why she had done that and she said Mummy told her it would remind me of what I was supposed to be doing right now. Back to the laundry I went.

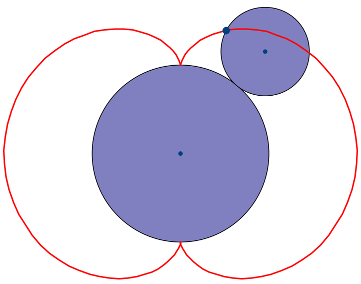

Figure 2: The resulting curve of the smaller Spirograph wheel being rolled around the larger.

After I had completed the laundry duties I set to thinking about this lovely natural occurrence. Three questions arose:

1. Why does it occur? (The reading thus far had not

told me.)

2. How might this be investigated by school students, at different levels, enabling them to make some sense of this phenomenon by learning why it occurs? (The teacher in me.)

3. What school-student mathematical research could grow out of observing the laundry-bucket nephroid? The teacher in me again.

Let's take each question in turn.

Why does it occur? (What more reading told me).

The laundry-bucket nephroid occurs because light rays from a source a reasonable distance away are *close* to being parallel to each other. In the process of reflecting from the circular wall of the bucket, the reflected paths of the many pseudo-parallel rays fall within a particular region - the brighter section of the bucket's base. It is the curved edge of this region that is the laundry-bucket nephroid. This curved edge is often referred to as the *envelope* of the family of reflected rays, *meaning the curve edge is tangent to each and every one of the reflected rays*. A nephroid would be produced if all the light rays being reflected were parallel.

In the field of Optics, an envelope is often called a *caustic*.

How might this be investigated by school students?

We can investigate the caustic phenomenon with students who know that for a reflection, the angle of incidence is equal to the angle of reflection. All we need is some paper, a sharp pencil, something round and a ruler. One way to proceed would be as follows. Illustrate the laundry-bucket phenomenon to the students in a few spare minutes of a lesson. Sometime later and preferably without telling them why, instruct the students to draw a circle and then draw one *incoming* straight line, from a point external to the circle, into the circle and reflect it off the *internal* surface that it meets. At this point I would be very happy with an *approximated* reflection. No constructions necessary. Have the students continue the process for *many* lines that are parallel to the first one drawn. Figure 3 illustrates the work of one student when given this task. The relative density of the lines illustrates why we get the curved edge, namely it is the border of two areas illuminated with different intensity.

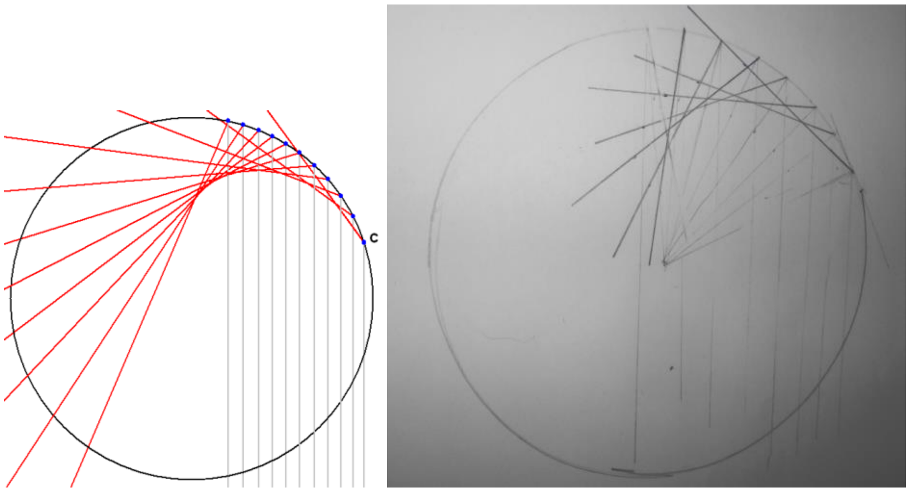

Figure 3: One student's effort of constructing a nephroid boundary.

Obviously there is some degree of estimation required when the reflection takes place! The student's ability to estimate will affect the quality of the output. So for some students in your class, their efforts may look very close to the laundry-bucket nephroid but for some it might not! However, by the end of the task, each student should have a fair idea how the laundry-bucket-nephroid comes into being. I would also hope that many had experience that *wow* moment of making the connection between the task and the bucket experience of some time earlier.

If the students have access to computer drawing software or geometry software, they could perform the same process electronically. This eliminates the variation due to estimation. Figure 4 illustrates the effect gained when the process is carried out using the software Geometry Expressions (GX) [3]. The process used is what I would term a *manual* electronic procedure. An incident and reflect ray, intersecting at C, were constructed and then C was moved to many different positions and the incident and reflected rays were copied and pasted.

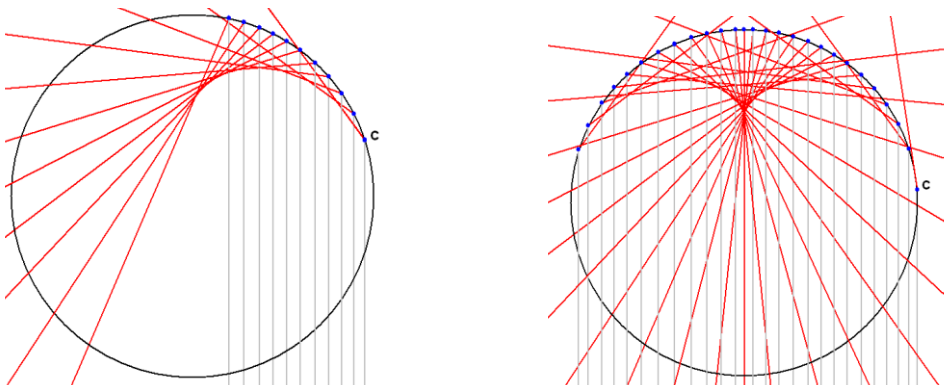

Figure 4: A parital and completed *manual* electronic version completed using Geometry Expressions.

<u>**An aside.**</u>

As we now know, the curved edge is the envelope of all the reflected rays. Using GX we can construct a *locus* of the reflected ray which results in GX displaying the envelope of the reflected rays (also known as the caustic) and in this case, we see a nephroid. See Figure 5.

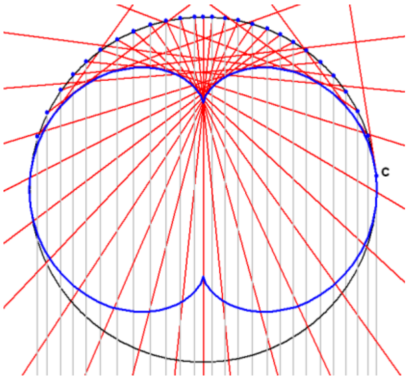

Figure 5: The nephroid, in blue, as constructed by GX.

It turns out there are many other ways to construct a nephroid (remember the Spirograph version). Take a look at http://en.wikipedia.org/wiki/Nephroid and you will be fascinated.

What school-student mathematical research could grow out of observing the laundry-bucket nephroid?

As time passed and the washing machine washed, two things struck me about the laundry-bucket nephropid. First, it had a very *pointy bit*. A nephriod has two. Second, even though the light source in the laundry was not producing parallel rays of light, a nephroid-like curve was produced and I noted that as one moved the bucket about the laundry floor (as I waited for the machine to tell me it was tme to do the hanging out) the curved edge changed shape somewhat and it even seemed in some cases there were two pointy bits.

Once the laundry was done a little more reading on the matter seemed in order. From the reading I learned that the laundry-bucket nephroid had a proper name. It is called the *circle caustic by reflection* or the *katacaustic* (the other kind of caustic, caused by refraction is called a *diacaustic*). Already I knew that if the light source was positioned infinitely far away from the reflective surface, then a nephroid resulted (as stated earlier), but I now had also learned that if the light source was on the circle circumference a cardoid was produced! Hope springs eternal, could have I been right after all? Where was that bucket first placed? I also learned that if the light source was at the centre of the circle then the caustic was the centre of the circle. But what about all the other possible light source positions?

All this reading had lead to more questions:

1. What is that pointy bit?
2. Where is the pointy bit relative to the light source?
3. Exactly how does the shape of the caustic change depending on the relative positions of the light source and reflective surface?
4. How was I going to answer all these questions?

More reading? Well more reading informed me that the pointy bit is called a cusp (but it is not that simple), but I found no answers to the other questions. What to do? The building of some sort of model, preferably one that could be manipulated, seemed like a good place to start. Earlier you saw a simple manual-electronic model build in GX. I decided to see if this could be developed further.

A little about Geometry Expressions.

GX is a symbolic geometry system (SGS). It has features similar to commonly known interactive geometry systems (IGSs), but is fundamentally different. As with IGSs, we can create geometric models by simple clicking and dragging operations. However, unlike IGSs, we can specify symbolic relationships between these geometric entities. For example, Figure 6 shows a GX model of a light ray (CB) emanating from a point C distance *a* units from the center A of a circle with radius AB. Along with defining the distance AC to be *a*, the length of segment AB is defined to be *r*, and the angle ACB to be θ.

In addition to being able to define the properties of an object in a symbolic manner, the software can compute the symbolic description of properties of geometric objects. For example, in Figure 6, we have asked GX to compute the size of angle CBA. The result provided? $\arcsin(\frac{a\sin\theta}{r})$.

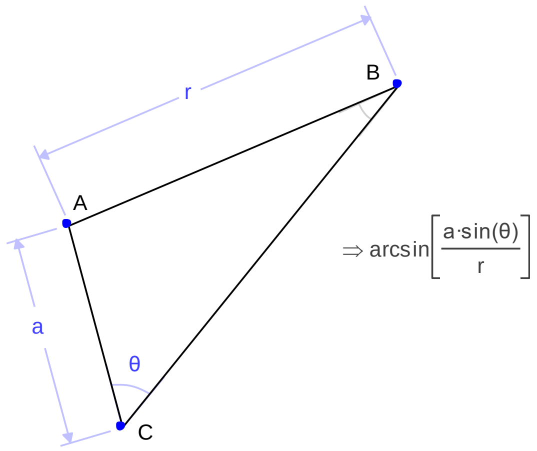

Figure 6. The symbolic result for angle CBA, given AC = *a*, AB = *r*, ACB = θ.

Building an interactive model of the circle caustic – where is the pointy bit and how does the shape change?

A model of the circle caustic, built in GX, can be seen in Figure 7. The light source, C, produces rays in all directions. This *family* of rays is represented by CB where B, the point of reflection, is a point on the circumference of the circle with centre O and radius 1 unit. The 'progress' of point B around the circle from the point (1,0) is described by the size the angle OB makes with the positive section of the x-axis and is defined to be *t* radians. BD represents the family of reflected rays. BD is produced by reflecting CB in the tangent to the circle at B. The circle caustic (in red) is produced automatically by GX.

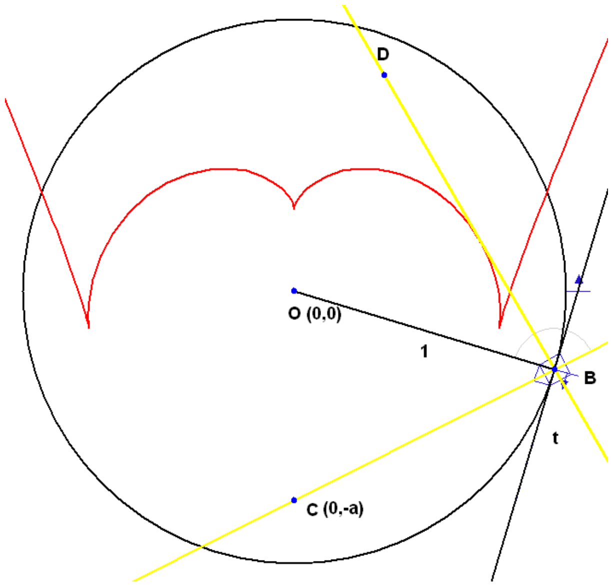

Figure 7: Geometry Expressions model of an incident beam of light CB reflected in a cylinder.

Given this is a dynamic model, we can investigate the effect on the position of the cusps and the shape of the circle caustic by varying the position of the light source C. Figure 8 shows the result for three different positions of the light source C. This is achieved by either dragging the point C, or by changing the value of the variable *a*.

Surprisingly, our model suggests some rather dramatic changes to the shape of the caustic that was not apparent in the base of the laundry bucket. It even suggests more pointy bits appear in some cases.

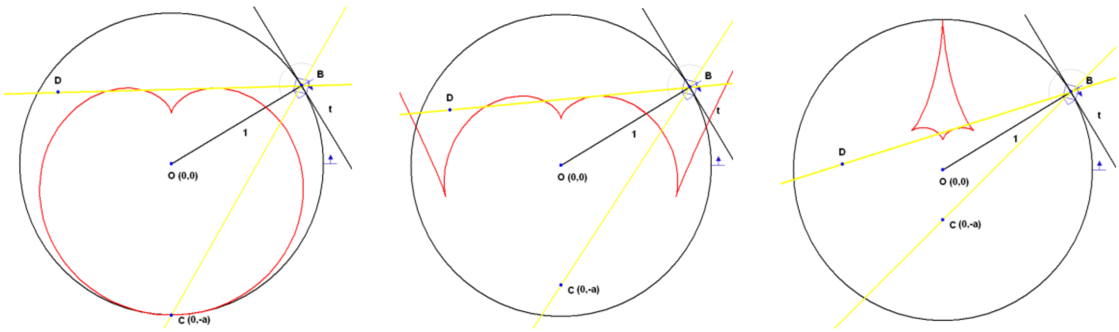

Figure 8. (a) $a = r$, (b) $a = 0.8r$, (c) $a = \frac{1}{3}r$.

Checking the model with the real thing.

Fuelled by the shock of the different and intriguing shapes, it was obviously necessary to see if such shapes were actually produced by real light. A suitable apparatus to investigate this is shown in Figure 9. This is a Perspex cylinder (actually a flower pot) with a disc of circular graph paper positioned at its base. The circular graph paper has 10 concentric circles with constant differences in radii such that it marks off the cylinder's base in tenths of the cylinder radius. A small LED lamp was used as the light source. The lids from bulk CD containers would make a good substitute for our flower pot.

Figure 9: Apparatus for studying caustic curves with a light source inside the reflecting cylinder.

In figure 10 we see the actual caustic shapes produced when the light source is at the circumference of the cylinder, when the light source moves closer into the center, and when the light source moves even closer to the center. Imagine that, the model seems to be correct.

(a) **(b)** **(c)**

Figure 10: Caustic curves observed with (a) the light at the circumference of the cylinder, (b) the light source interior to the cylinder and (c) with the light source closer to the centre of the cylinder.

<u>Locating the cusps.</u>

The location of the cusps can also be thought about as which incident light ray passes directly through the cusp. We can experiment with our model, by moving point B to gain some idea about this, considering the value of *t* as we move. In Figure 9 we see the case for $t \approx \frac{\pi}{2}^c$ and $t \approx 6^c$; the cases where the incident ray seems to coincide with the cusps.

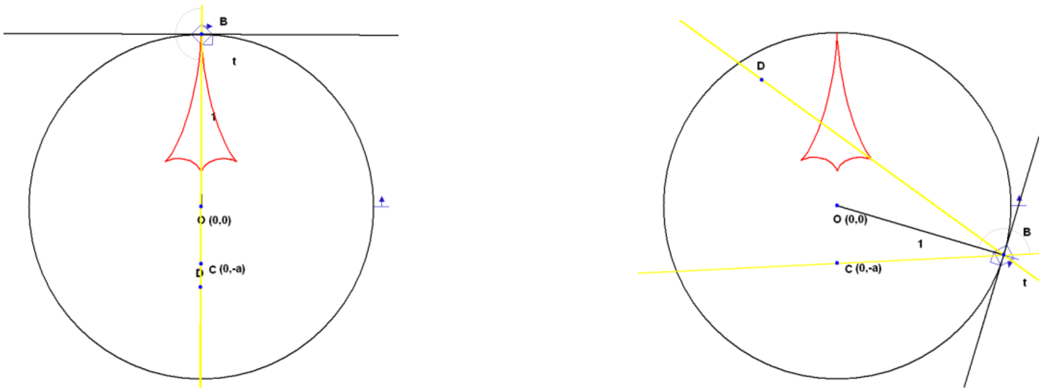

Figure 9: Our model where $a = \frac{1}{3}r$ and (i) $t \approx \frac{\pi}{2}^c$ and (ii) $t \approx 6^c$

Another way to consider this is to have a point on the caustic itself. In Figure 10, point E has been placed on the caustic, but most importantly is has been located *t* units along the caustic. Therefore its position is tied to the position of B. Now I would dearly love to know the cartesian coordinates of E in terms of *t*. Given the symbolic capabilities of GX, we can query it for the coordinates of E and it replies (as seen in Figure 10):

$$\mathrm{E}\left(\frac{2a^2\cos^3 t}{1+2a^2+3a\sin t}, \frac{a(2+3a\sin t+a\sin 3t)}{2(1+2a^2+3a\sin t)}\right)$$

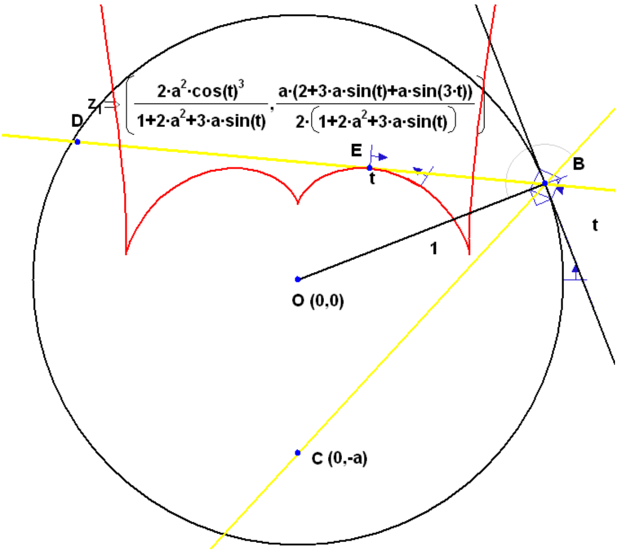

Figure 10: Our model with point E added and its coordinates in parametric form.

Working with the model – cusp locations.

Image (a) and (c) of figure 4 illustrate special cases. Image (a) illustrates the case where the light source is at the circumference of the cylinder while image (c) illustrates the case where the upper-most cusp is at the circumference.

Look at image (a) of figure 4 and estimate the distance of the cusp from the centre of the circle as a fraction of the radius. (remember each ring on the paper is 1/10 of a radius).

Figure 11 shows the model when we have positioned the light source (C) at the circumference by specifying its coordinates to be (0,-1). We have also put a point on the curve at parametric location t. Note that (i) its coordinates are identical to the parametric equations of the curve, and (ii) it is the point of contact of the ray reflected from B (which is at angle t on the circle).

This correspondence lets us determine the parametric location of the cusp. We can either eyeball this by dragging the point B on the circumference, or make a symmetry based argument to suggest that the cusp is located when the incident ray is vertical, that is at an angle of $\pi/2$. (We will make a more precise – but necessarily more complicated – defense of this assertion below)

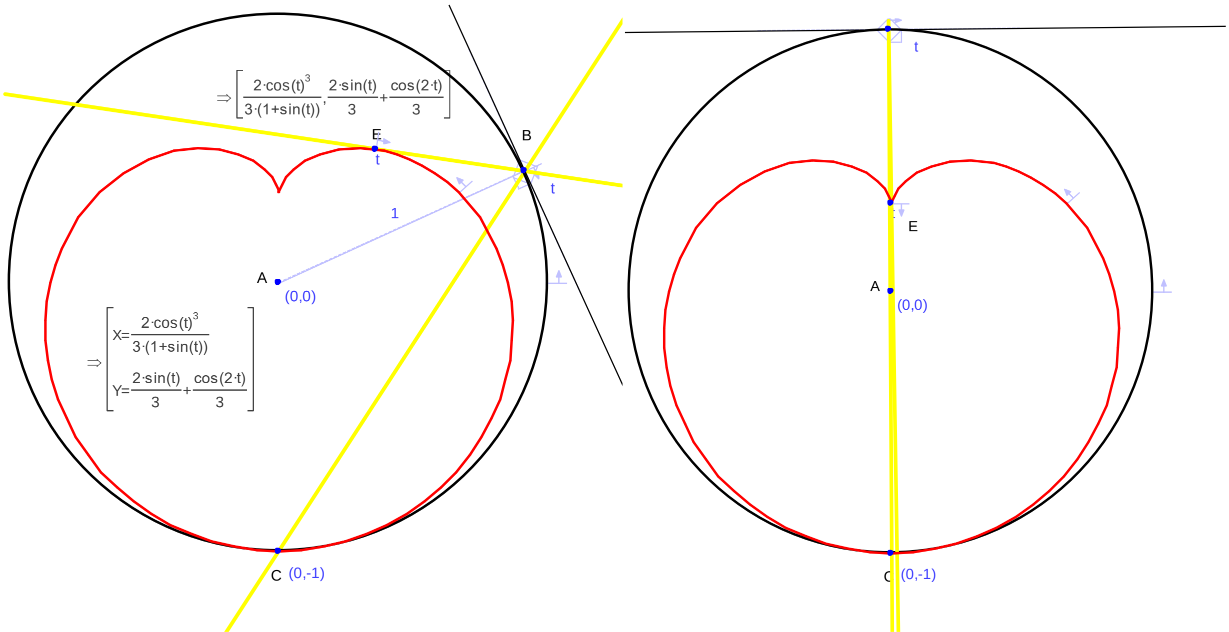

Figure 11.

If we substitute t= π/2 into the parametric equations of the caustic we find:

$$x = \frac{2\left(\cos\frac{\pi}{2}\right)^3}{3\left(1+\sin(\frac{\pi}{2})\right)} = 0$$

$$y = \frac{2\sin\frac{\pi}{2}}{3} + \frac{\cos\pi}{3} = \frac{1}{3}$$

How does this match your observation from image (a) in figure 4?

As the light source moves from the circumference towards the center of the cylinder, the caustic curve is no longer fully contained in the cylinder (image (b) of figures 4 and 10). However, at some point, it re-enters to give the shape in image (c) of figures 4 and 10.

Look at image (c) of figure 4 and estimate the distance of the light source and the lowest cusp from the centre of the circle as a fraction of the radius.

In Geometry Expressions, with the coordinates of C defined to be (0,a) we can drag C towards the center of the circle until the second central cusp appears. Again, dragging B leads us to believe that the central cusps are the result of rays reflected from B when t= π/2 and 3π/2 . We could compute the coordinates for the points on the caustic when t= π/2 and 3π/2 by substituting into the general equation for the curve (seen in figure 10). Alternatively, Geometry Expressions allows us to specify the parametric location of a point on a curve. Hence one can simply create points at parametric locations t = π/2 and t = 3π/2 on the curve and ask for their coordinates; as seen in Figure 12.

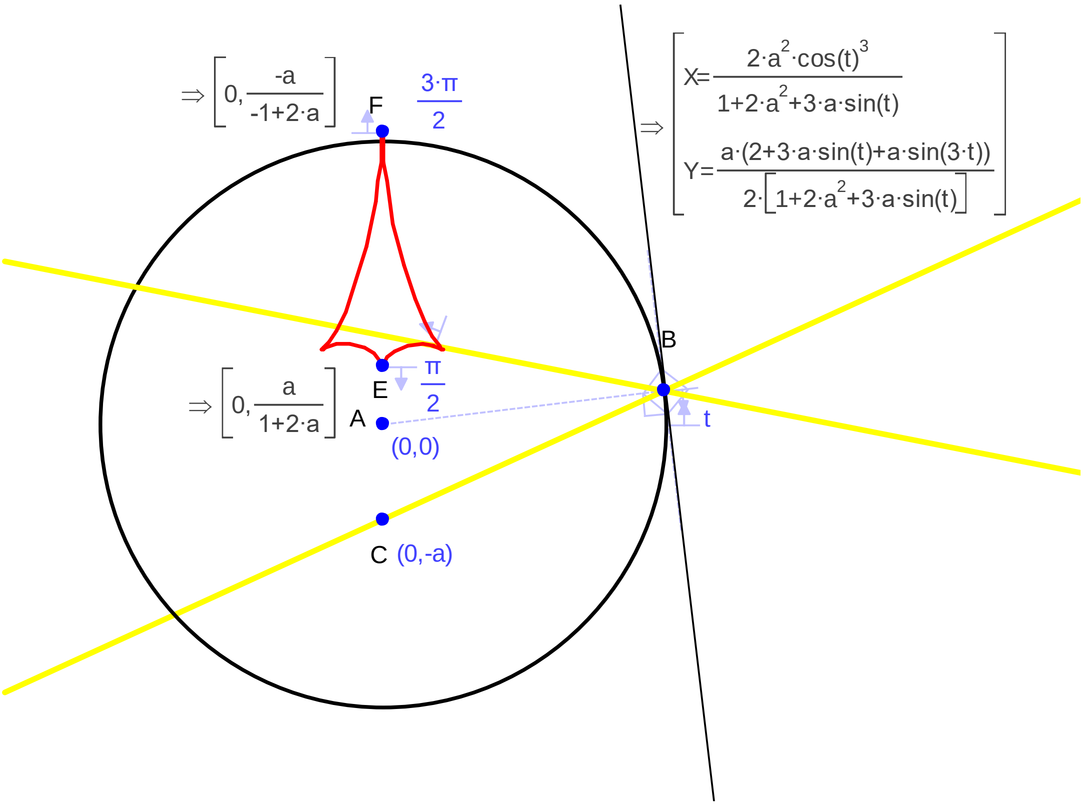

Figure 12: Points E and F are put on the curve and constrained to be at parametric locations π/2 and 3π/2 and their coordinates computed, in terms of a.

For F to lie on the circumference, we need:

$$y = \frac{-a}{1+2a} = 1$$

Solving for a yields:

$$a = -\frac{1}{3}$$

This tells us the light source would be located at one-third of the circle's radius from its centre.

To find the location of the lowest cusp we can substitute as follows:

$$y = \frac{a}{-1+2a} = \frac{-\frac{1}{3}}{-1-\frac{2}{3}} = \frac{1}{5}$$

How closely does this correspond to your observations from figure 4c?

When the source is outside the circle.

In figure 1, the light source is many radii away from the center of the cylinder. In this case, where, approximately, will the cusp lie?

Assuming a is effectively infinite. Our cusp will now have location:

$$\lim_{a\to\infty} \frac{a}{1+2a} = \frac{1}{2}$$

In figure 1, therefore, the cusp should appear approximately half way between the center of the ring and the circumference. How does it look?

Observing the angle of incidence.

With the light source inside the circle, we note that the angle of incidence is 0 radians for t= π/2 and 3π/2. But what about for values of t in between π/2 and 3π/2? We can ask Geometry Expressions to measure the numerical value of angle ABC, (seen in figure 13) and then drag B to investigate this. Doing this leaves us with the conjecture that: (1) the angle increases to a maximum and then decrease to zero and that the maximum occurs when angle ACB is a right angle, (2) that this ray corresponds to the cusp at E (as seen in image (b) of figure 13).

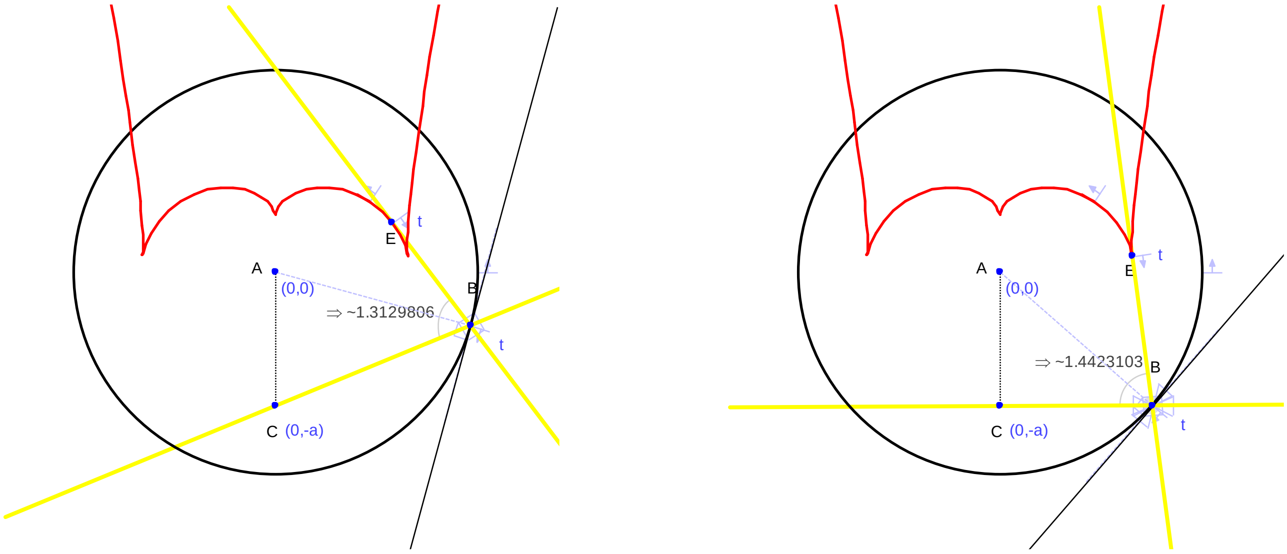

Figure 13: Moving B around the circle, we observe that the maximum angle between incident and reflected rays occurs where CB is perpendicular to CA. The reflected ray also appears to pass through the cusp of the caustic.

To prove the first part of our conjecture we consider triangle ABC. Using the sine rule we have:

$\dfrac{\sin C\hat{B}A}{dAC} = \dfrac{\sin A\hat{C}B}{1}$ and so $\sin C\hat{B}A = dAC \times \sin A\hat{C}B$

Now angle CBA can not exceed a right angle and so for it to be a maximum, $\sin A\hat{C}B$ must be a maximum and so angle ACB must be a right angle.

An alternative to this argument, which uses more mathematics, but less insight, is to use Geometry Expressions to generate a formula for the angle (figure 5), then a CAS to differentiate and solve. A further alternative, which uses less mathematics (geometry only, no trig.) but more insight, is presented in Appendix A.

To prove the second part of our conjecture, we appeal to a result of differential geometry which states that cusps occur where the derivative of x and y with respect to t simultaneously vanish. While daunting to perform by hand, with a CAS it is an easy task to differentiate the expressions for X(t) and Y(t) (see figure 12), then solve for t. The process is made easier by the fact that you can copy and paste the equations directly from Geometry Expressions into your CAS system without retyping.

Note that the solution given by Maple is a 2 agument arctan. Arctan(y,x) is defined to be $\tan^{-1}(y/x)$ where y is non zero and $\pi/2$ where y=0. The solution is a value for the parameter t, which is both the parametric location of point G on the circle, and the parametric location of the point of contact between the reflected ray and the caustic curve. Once the solution is found, it can be used as the parametric location of a point on the curve in Geometry Expressions (figure 14), the software can tehn generate the cartesian coordinates of the point.

We note first that the y coordinate of point G is the same as the y coordinate of point C. Hence CG is horizontal and angle ACG is right. Less obvious is the fact that point F is the image of C under reflection in the line AG. This can be verified by hand, or using Geometry Expressions (fig 15).

While the algebraic determination using our symbolic geometry and CAS technology is clear, it would be nice if we could come up with an exclusively geometric explanation for this geometric phenomenon. Here is my attempt.

Regardless of where G lies on the circumference of the circle, the reflected ray will pass through C', the image of C under reflection in AG. Can you prove that the locus of C' is the circle centered at A which passes through C?

Now if you look at figure 16, you can see that although C' is on the reflected line, it is not necessarily on the caustic. In fact the caustic is inside the locus of C' part of the time and outside part of the time, with the division between inside and outside happening at the cusp.. just when GC' is tangent to the locus of C'. (Another way of thinking of this is that this is the instant when the reflected ray is stationary, as C' is moving along the direction of the ray). By symmetry, if GC' is tangent to the circle centered at A through C then GC is also tangent, and angle ACG is right.

When it comes to differential geometry, I have to admit that my intuition is anything but infallible, so feel free to treat the above intuitive argument with a pinch of salt.

$$X := \frac{2\cos(t)^3 a^2}{1 + 2a^2 + 3\sin(t)\,a}$$

$$Y := \frac{(2 + 3\sin(t)\,a + \sin(3\,t)\,a)\,a}{2\,(1 + 2a^2 + 3\sin(t)\,a)}$$

```
> solve({diff(X,t)=0,diff(Y,t)=0},t);
```

$$\left\{t = -\frac{1}{2}\pi\right\}, \left\{t = \frac{1}{2}\pi\right\}, \{t = \arctan(-a, \mathrm{RootOf}(a^2 - 1 + \_Z^2))\}$$

```
> allvalues(arctan(-a, RootOf(a^2-1+_Z^2)));
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$$\arctan(-a, \sqrt{1 - a^2}),\ \arctan(-a, -\sqrt{1 - a^2})$$

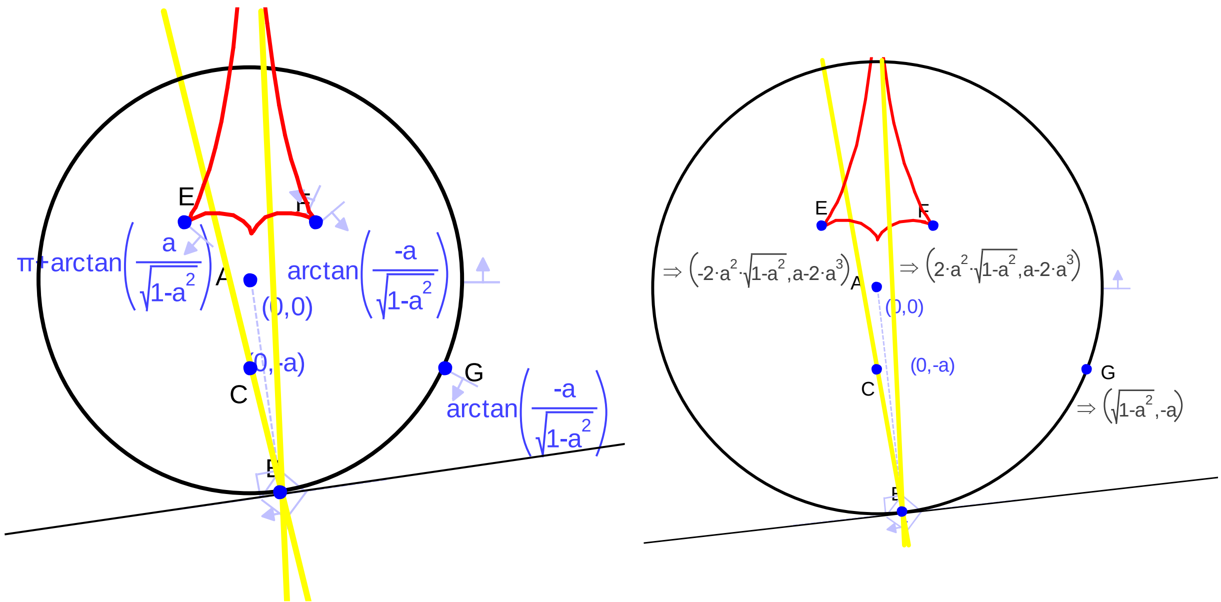

Figure 14: Derivation of cusp locations using Maple. Parametric and cartesian locations for cusp points and corresponding reflection point are shown.

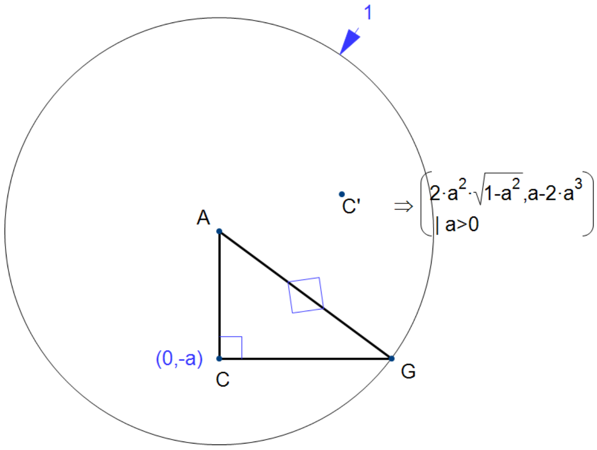

Figure 15: point C' is the image of C under reflection in the line AG. We see that it has the same coordinates as the cusp shown in Fig 14.

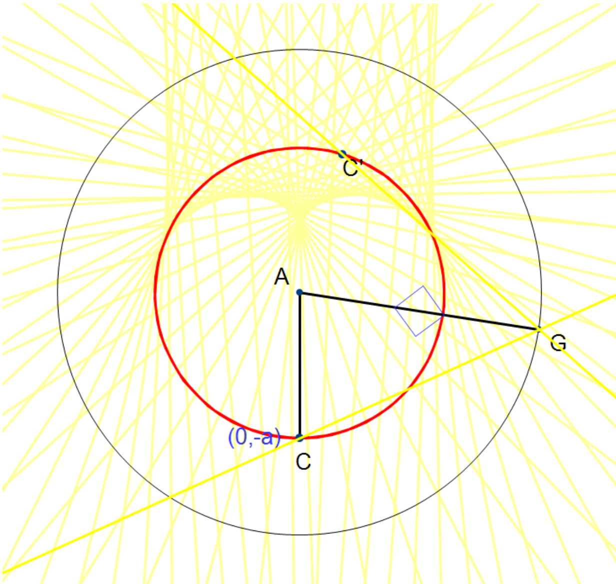

Figure 16: The locus of C' as G runs along the circle is a circle centered at A through C. Although the reflected ray passes through C', C' does not necessarily lie on the caustic. When ACG is right, GC and GC' are tangent to the locus of C', and the reflected ray is stationary, forming a cusp.

When two cusps sit on the *x*-axis.

When do 2 cusps sit on the x-axis? For this to happen, the y-coordinates of the cusps must be 0. To find the corresponding location for the light source, we need to solve for a:

$$a - 2a^3 = 0$$

This can be factored:

$$a(1 - 2a^2) = 0$$

And solved:

$$a = 0,\ a = \frac{1}{\sqrt{2}}$$

Can you interpret the solution geometrically?

Conclusion.

We have investigated the locations of cusps in the circle catacaustic curve. The caustic curve is easy to create and observe in real life using simple apparatus. Mathematical analysis without computer would involve the tools of classical differential geometry which are substantially beyond the high school curriculum. However, use of technology, specifically in the form of a new Symbolic Geometry program makes the mathematical modelling of this curve accessible at an Algebra 2 or Precalculus level.

The ability to mix algebraic and geometrical representations in the same computer model allows the modelling of this physical phenomenon to provide a rich case study in parametric curves. The meaning of the parameter in a parametric curve can be a tricky abstract concept. The challenge of positioning a point at an identifiable landmark on the curve make this a compelling example to explore the relationship between parametric location and coordinate location.

If we introduce, from differential geometry, the notion that the cusp corresponds to a parametric location where the derivatives of both coordinates vanish, then determination of cusp location is a matter of differentiating and solving. The expression complexity is such that use of a CAS is justified. However the solutions are sufficiently simple to admit a direct geometrical interpretation.

This, then is an example where technology enables mathematical modelling of a rich problem domain to be feasible in a high school setting. Specifically, the seamless interplay between geometry and algebra embedded in a symbolic geometry system, along with the easy communication between symbolic geometry and CAS alowed the dual geometric/ algebraic nature of the model to be exploited to the hilt.

References

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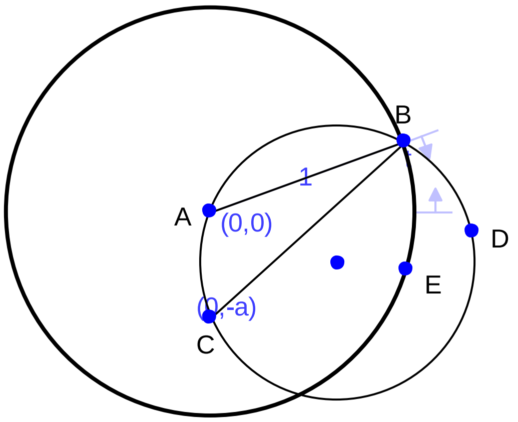
Bookmarked pages in FF in PC.

http://jwilson.coe.uga.edu/Texts.Folder/Envel/Envelopes.html

http://thesaurus.maths.org/mmkb/entry.html?action=entryByConcept&id=3240&msglang=es

Appendix A

We showed above, using trigonometry, that the maximum angle of incidence occurs when the incident ray is perpendicular to the line joining the light source to the center of the circle. A nice geometric argument which proves this result is based on the observation that all the points D such that ADC = ABC lie on the circumcircle of ABC (figure 17). If this circumcircle intersects the original circle in two places, and if E lies on the portion of the circumference of AB cut off by the circumcircle then AEC>ABC. Hence, ABC can only be maximal if the circumcircle to ABC is tangential to circle AB (fig 17b). In this case it is easy to see that AB is a diameter of the circumcircle and hence the angle ACB is right.



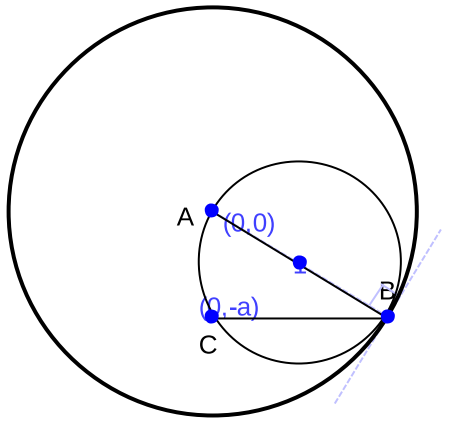


Figure 17: (a) E is inside the circumcircle of ABC, hence angle AEC < angle ABC. (b) The circumcircle of ABC is tangent to the original circle. This corresponds to a maximum of angle ABC.

To explore the relationship between the source position and the number and position of the cusps, we need to develop a model of this phenomenon. Mathematically, the caustic is the curve which is tangent to the entire family of reflected rays [1]. Such a curve is called the envelope of the family of lines [2].

Envelopes are properly a part of differential geometry, which students will not be exposed to until late in their college careers. However, new symbolic geometry technology can assist us to study aspects of these fascinating curves at a level of Algebra 2 or pre-calculus and in the process consolidate understanding of concepts taught in high school.