

MODELING REAL-WORLD CALCULUS PROBLEMS WITH SYMBOLIC GEOMETRY SOFTWARE.

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Digital technologies have been introduced to mathematics education over two decades ago. With the rapid development of computer technologies, its influence on teaching mathematics at all educational levels becomes more and more significant. Computer algebra systems (CAS) and dynamic geometry software (DGS) have been widely adopted in the last 20 years. Symbolic geometry software, *Geometry Expressions*TM, is relatively. It is a dynamic, constraint-based symbolic geometry system that allows students to model the geometry to be expressed algebraically and then solved automatically in a computer algebra system. *Geometry Expressions* provides a mathematical tool that produces outputs in a symbolic form as algebraic expressions involving the input parameters. Mathematical expressions can be copied back and forth between the symbolic geometry system and a CAS thus making it an interesting new tool for exploring mathematics and solving problems, *Geometry Expressions* also provides teachers with the opportunity to generate apps that could be used on different mobile platforms or with an internet browser on the computer, thus giving students an opportunity for independent out-of-school explorations of mathematics. The apps generated with the *Geometry Expressions* provide numeric and graphical outputs only. Thus, apps could be used by the students for an initial exploration of a problem or a concept, maybe even prior to the lesson where this problem or concept is introduced. Then, more rigorous analysis of the problem with symbolic outputs and CAS could become the next stage of the investigation. In this paper, we discuss three examples of using *Geometry Expressions* for modeling real-world and/or complex calculus problems, as well as the process of app generation and teaching strategies using apps.

Keywords: calculus, modeling, symbolic geometry system (SGS)

Introduction

Digital technologies have been introduced to mathematics education over two decades ago. With the rapid development of computer technologies, its influence on teaching mathematics at all educational levels becomes more and more significant. However, the mathematics education community is still searching for answers to questions addressing three main issues: whether technology should change the focus of the mathematics curriculum, whether technology use changes how students conceptualize mathematics, and whether the benefits of technology outweigh the costs (Risser, 2011; Weigand, 2014). Over a decade ago, the National Council of Teachers of Mathematics (2000) categorically stated that technology should be an integral part of the educational experience: “The existence, versatility, and power

of technology make it possible and necessary to reexamine what mathematics students should learn as well as how they can best learn it. In the mathematics classroom every student has access to technology to facilitate his or her mathematics learning under the guidance of a skillful teacher” (p. 25). Roschelle *et al.* (2000) identified four fundamental characteristics of how technology can enhance what and how students learn in the classroom: active engagement, participation in groups, frequent interaction and feedback, and connections to real-world contexts. Studies show that incorporating application projects and increasing the use of technology in the classroom increase the well-roundedness of students’ education and improve their performance in more advanced courses at both, secondary and post-secondary levels (Schreyer-Bennethum & Albright, 2011).

Many mathematicians and mathematics educators view mathematics as an experimental science. (Cnop, 2003; Lyublinskaya, 2012; Ryzhik, 2011). Cnop (2003) links experimentation with the exploration of “real-world” problems reporting that, “Thanks to the introduction of technology this functionality is again possible. . . . Students can rephrase (simple) real-world problems, leave computing to the software and focus on the qualitative analysis of the result and understanding.” (p. 2). A growing number of international studies have shown that Computer Algebra System (CAS-based) instruction has the potential to positively affect the teaching and learning of mathematics at various levels of the education system (Artigue, 2002; Lavicza, 2006; Pierce & Stacey, 2004). Using CAS in college level mathematics provides an experimental laboratory for students in which they could explore mathematics objects. A review of 204 papers on CAS-based instruction identified the three most common uses of CAS in the classroom: experimentation or exploration in which an individual or small group of students use the software to do mathematics; visualization in which students use CAS as a tool for creating or viewing or modifying graphical representations of mathematical functions and objects; and analyzing real-world, complex problems in which students use CAS to model and interpret mathematical phenomena. (Buteau *et al.*, 2010).

Similarly, dynamic geometry software (DGS) has been extensively used for teaching mathematics at various educational levels. DGS allows students to explore mathematical objects visually and dynamically, formulate conjectures and intermediate results, examine special cases, and generate new ideas based on their observations. Wide exposure of DGS for millions of creative minds, both teachers and students, has resulted in a plethora of pedagogic uses for this technology (DePeau & Kalder, 2010; Gawlick, 2002; Lyublinskaya & Ryzhik, 2008; Olivero & Robutti, 2007; Scher *et al.*, 2004; Todorova, 2012). DGS enables students to choose various quantities as independent variables, providing different dynamic representations of a situation and the exploration of properties of mathematical objects. Research shows that DGS may foster dynamic thinking for students solving mathematical problems. Data collected in the study by Arzarello *et al.*, (2012) suggested that the dynamic features of technology supported development of conjectures, and their validation or refutation. In another study, Verzosa *et al.* (2014) documented how DGS tools may be used for teaching the rigorous foundations of calculus and showed how a combination of dynamic tools and guide questions allowed students to develop their understanding of calculus ideas.

Computer algebra systems (CAS) and dynamic geometry software (DGS) have been widely adopted in the last 20 years. Symbolic geometry software, *Geometry Expressions*TM, is relatively new (Todd, 2007). It is a dynamic, constraint-based symbolic geometry system that allows students to model the geometry to be expressed algebraically and then solved

automatically in a computer algebra system. *Geometry Expressions* provides a mathematical tool that produces outputs in a symbolic form as algebraic expressions involving the input parameters. Mathematical expressions can be copied back and forth between the symbolic geometry system and a CAS thus making it an interesting new tool for exploring mathematics and solving problems. (Lyublinskaya & Funsch, 2012; Ottman, 2010, 2012; Todd *et al.*, 2010; Todd & Wiechmann, 2008). In calculus, it enables more extensive open-ended explorations / investigations than are possible in a traditional course (Lyublinskaya *et al.*, 2010).

Usually access to educational software for students is limited to the school computer lab, unless students are required to purchase software for home use. This prevents students from computer experimentations and explorations at home. *Geometry Expressions* provides teachers with the opportunity to generate apps that could be used on different mobile platforms or with an internet browser on the computer, thus giving students an opportunity for independent out-of-school explorations of mathematics (Lyublinskaya, 2012; Ottman & Lyublinskaya, 2013; Todd, 2013). With app generation, the teacher has an opportunity to define ranges of parameters, ways that parameters can be manipulated, and what information will be available to the students (like measurements and calculations). By nature all apps are numeric based and will not display any symbolic outputs. Thus, the apps could be used for exploration of major calculus concepts, generation of conjectures, and finding approximate numerical answers to given problems. The use of apps can also be extended to practicing skills. The apps generated with the *Geometry Expressions* provide numeric and graphical outputs only. This enables another approach to teaching mathematics. Apps could be used by the students for an initial exploration of a problem or a concept, maybe even prior to the lesson where this problem or concept is introduced. Then, more rigorous analysis of the problem with symbolic outputs and CAS could become the next stage of the investigation. Finally students could be asked to verify all work by hand in order to practice necessary skills. Many collections of interactive mathematics apps generated with *Geometry Expressions* can be found on the Euclid's Muse website (<http://euclidsmuse.com/>).

In this paper, we will discuss three examples of using *Geometry Expressions* for modeling real-world calculus problems as well as the process of app generation and teaching strategies using apps. HTML5/Java apps generated for the examples in this paper can be accessed on the Euclid's Muse website under collection "Modeling calculus problems".

Focus on calculus

We know the difficulties in motivating many students to develop the habits of mind and critical thinking skills necessary to thoroughly understand the concepts of calculus. Many calculus problems can be visualized with the help of geometric representation. Major calculus concepts, such as derivative and integral, have a clear geometric meaning. A function can be represented geometrically by its graph, the derivative of the function is visually represented by the slope of the tangent line, and the definite integral of the function is the area under the graph of that function. Optimization problems that deal with areas and perimeters of geometric shapes provide solutions to geometric problems. A geometric image helps in setting up primary equations in related rates problems. In general, visual representation can help with the interpretation and solution of an analytical problem, sometimes providing a more effective problem solving method.

Now, let us illustrate the use of a symbolic geometry system applied to three problems adapted from *Student Research Projects in Calculus* (Cohen *et al.*, 1991). The first problem, "The Bicycle Race", focuses on the concepts of derivative and velocity. The second problem,

“A Greenhouse Extension” is an optimization problem, and the last problem, “Exploring Unchartered Waters” focuses on the second fundamental theorem of calculus. The first two problems model more or less real-world situations, while the third one is purely contrived. Even with an unrealistic story line in the third problem, students enjoyed working on all three problems (Cohen, *et al.*, 1991). We selected these problems as they help demonstrate various features of the symbolic geometry software when used for student investigations of calculus concepts. All three problems are appropriate for students at the upper secondary or college entry level.

The Bicycle Race (Cohen *et al.*, 1991, p. 94)

In this problem, a bicycle racer is moving at a constant speed. As she passes her support car, the car starts accelerating from a dead stop in order to catch up with the bicyclist to provide her with a cold drink. The equation of motion for the support car is given by the function $\frac{1}{3}(10t^2 - t^3)$. Students are asked to analyze the situation algebraically and graphically and determine the speed of the bicyclist to assure that speeds match the moment the support car catches up with the racer. Additional questions ask students to analyze what happens when the speeds do not match.

In order to model this situation in *Geometry Expressions*, students can draw graphs that depict the distance traveled by the bicyclist and by the car plotted on the same axes. We use x for time and y for distance traveled. This is the time to ask students questions about these graphs.

- How does the graph illustrate the fact that the racer and her support car have the same velocities at the same location?
- What additional graphical information do you need in order to illustrate this fact?

The velocity of the racer is the constant k . The distance traveled by the racer is represented by the linear function $y = kx$. In the software, students can plot this function and use a slider to change the value of k . The distance traveled by the support car is defined by the cubic function

$y = \frac{1}{3}(10x^2 - x^3)$. At any instant in time, the instantaneous velocity of the car is defined as the slope of the tangent line at a corresponding point on the graph. Analyzing the situation, students should arrive at the decision to draw a tangent line to the graph of the cubic function. When the tangent line is drawn, the software constructs the point of tangency (point A in Fig. 1). For further analysis, we will constrain the x -coordinate of point A as t and calculate the slope of the tangent line. As shown in Figure 1 the CAS embedded in the software leads to the expected expression for the derivative of the function at $x = t$. Students can move point A along the curve to observe the changes in the slope.

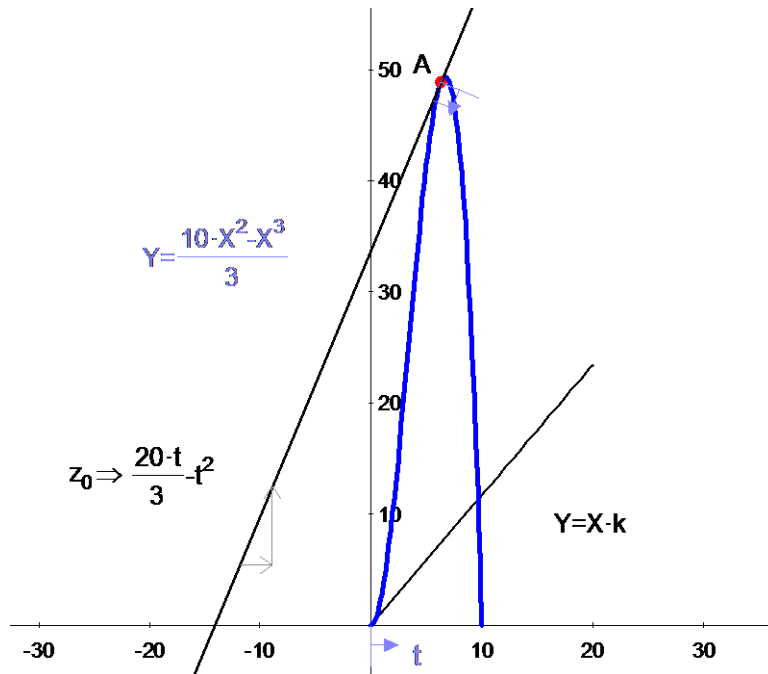


Fig. 1. Modeling the Bicycle Race problem in the *Geometry Expressions*

Considering the two conditions for the racer and the car (same position and same velocity), students can now illustrate this situation graphically by finding a point on the curve where the tangent line to the cubic function and the line representing the distance traveled by the bicyclist coincide. The software also provides numeric values (approximate) for k and t at this point. Students can use CAS or paper and pencil to find the exact solution to the problem and to justify their graphical interpretation. The dynamic nature of this computer model allows students to answer follow up questions given in the problem:

- How many times would the car and the bike meet if the racer were going faster than velocity found above? Slower?

The approach considered above can be used with a whole class, small groups or as an individual activity with students working on computers with the installed software in the school computer lab. This problem can also be assigned as an independent homework project. In this case, students might not have access to the software. For this situation, let us generate an HTML5/Java applet for student investigation. The model can be exported from *Geometry Expressions* in the form of a JavaScript app and embedded in an html page. This app can be put on the teacher's website, or emailed to students for them to open in their browser on whatever device they use for their homework. In this applet we will allow students to change parameter k using a slider and to drag point A along the curve. In addition, students will be able to see the numeric values of k , t , and the slope of the tangent line to the cubic function (Fig. 2).

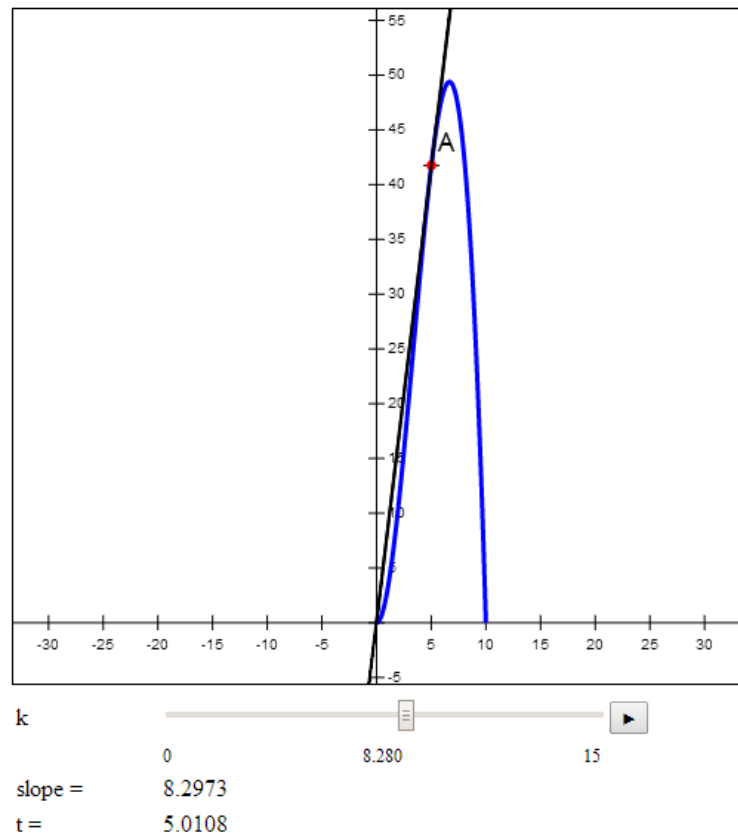


Fig. 2. Applet for investigation of the Bicycle Race problem

Using this applet, students can investigate the problem conceptually in order to develop a mathematical model for the given situation and find the approximate numerical answers to the questions posed in the problem. They can then solve the problem with paper and pencil, and compare the results of calculations completed by hand with the results provided by the applet.

A Greenhouse Extension (Cohen *et al.*, 1991, p. 106)

This is a classical optimization problem where students are asked to optimize the usable floor space of the greenhouse that their parents are building adjacent to their house. The floor space in the greenhouse is only considered usable if both parents can stand up in it. The roof of the greenhouse is the sloped glass of a given fixed size.

The first step for the students is to develop a geometrical model for this problem and to introduce all variables and constants in the problem. Here is one possible way to do it. We will use coordinate axes as the wall of the house and the ground level (Fig. 3). The segment AB represents the sloped glass, with point A on the wall of the house and point B on the ground. The point B can be moved along the x -axis and its location will determine the maximum usable floor space. Thus, the x -coordinate of point B is the independent variable. The side length of the glass roof is constrained as L , and since it is fixed, the numeric value of this parameter is locked. In order to represent the height of the tallest parent, we will draw a line parallel to the floor at a height h . Students can select the value of h individually according to the height of

their tallest parent. The intersection of this line with the segment AB defines the point where the “head of the parent” will touch the ceiling. Thus, the distance CD is the length of usable floor and it is the dependent variable. Mathematically we are looking for the maximum distance, CD, as a function of x . The software will produce a symbolic output for this distance in terms of all introduced parameters. (Fig. 3)

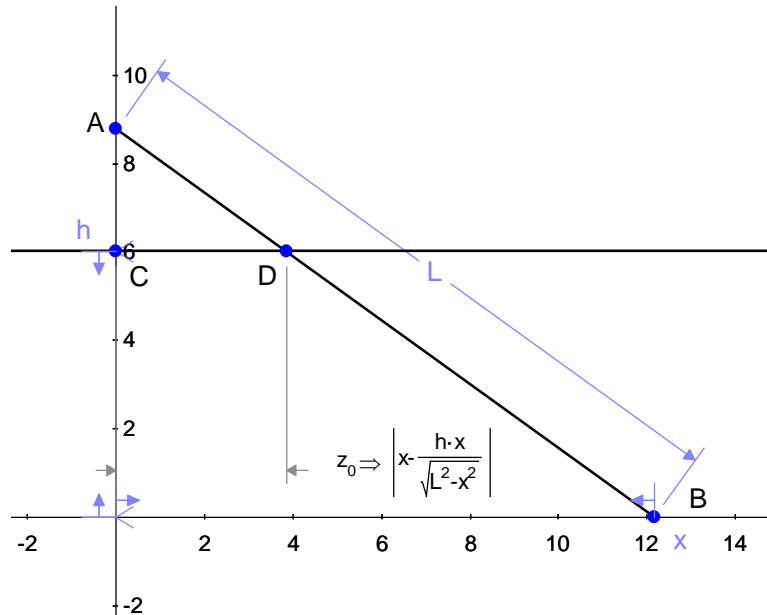


Fig. 3. Geometrical model of the Greenhouse Extension problem.

As you can see, the CAS provided students with the algebraic expression for the function to be optimized as long as they developed the correct geometrical model of the problem. Dragging point B along the x -axis can help students verify the maximum distance CD . At this point, the teacher might ask students to use the traditional approach of finding the maximum of the function by hand. *Geometry Expressions*, however, allows for a more visual approach by graphing the resultant function on the same axes, plotting the tangent line to the graph at a point E with the same x -coordinate as point B , and calculating the slope of the tangent line (Fig. 4). Moving point B along the x -axis demonstrates that the maximum of the function (which corresponds to the largest distance CD) is achieved when the tangent line is horizontal. Students can then solve for the derivative equal to zero, to complete the problem’s solution.

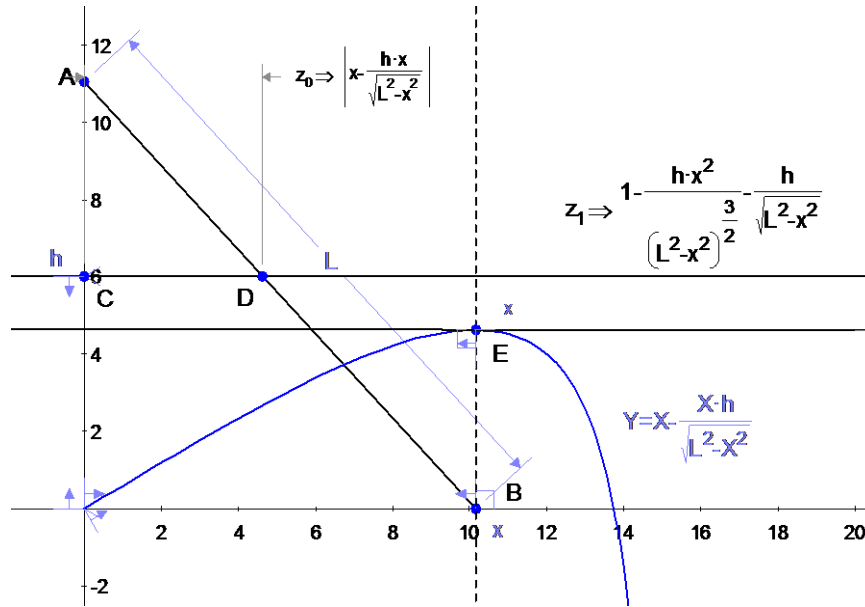


Fig. 4. Graphing approach to the Greenhouse Extension problem

Alternatively, the formula for the distance CD can be copied from *Geometry Expressions* and pasted into an external CAS program, to differentiate and solve the expression. Figure 5 shows the steps required to carry out this process with the TI-Nspire CAS.

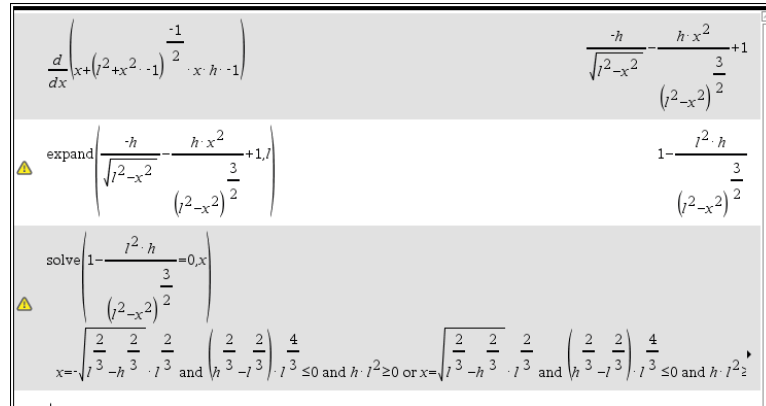


Fig. 5. Symbolic calculations for the Greenhouse Extension problem in TI-Nspire CAS software

While CAS provides full set of solutions for a given equation, students need to select the solution appropriate for the conditions of the problem, such as $0 \leq x \leq L$ and $h \leq \sqrt{L^2 - x^2}$.

Similar to the previous example, the geometric model for this problem could be exported as an applet (Fig. 6). In this applet students enter the height of their tallest parent in the textbox. Point B is made draggable along the x -axis. The student can drag point B and observe how the area varies with its placement. She can change the height of her parents and observe how this impacts the solution. Using this applet allows students to explore the situation and to make a conjecture for the maximum. They will also find numerical values for the

maximum usable floor space and the position of the sloped glass given its length and the height of the tallest parent. However, the use of the applet does not provide symbolic outputs, so students will have to use geometry to arrive at the algebraic expression for the function to be optimized. Thus, in generating the applet the teacher might consider making additional constructions and indicating coordinates of all points to help students to see the relationships between variables.

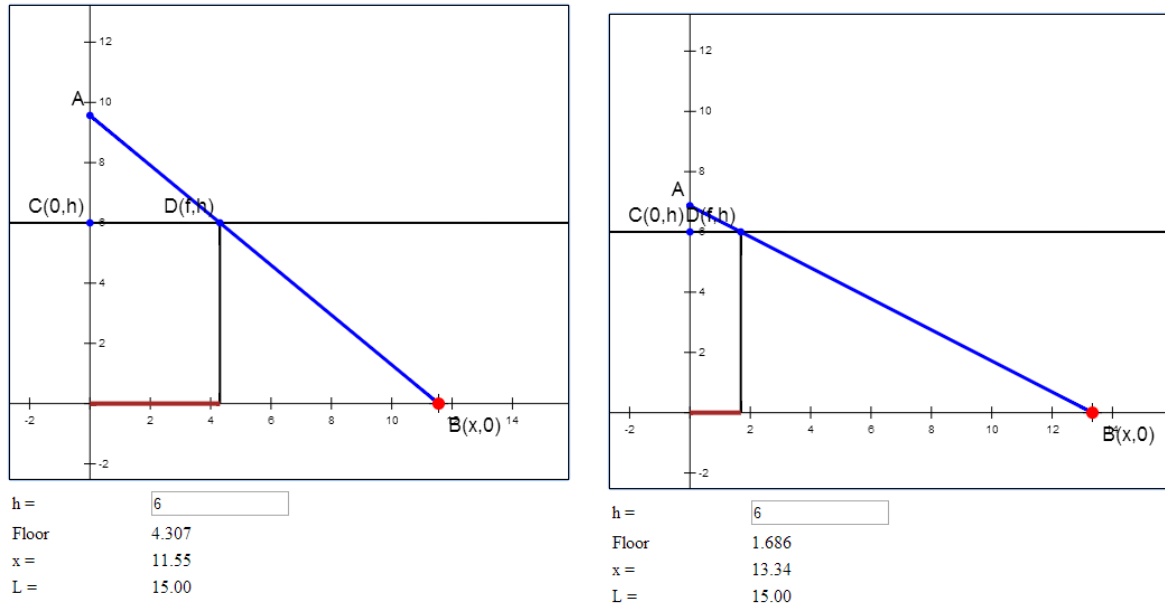


Fig. 6. HTML5/Java applet for student investigation of the Greenhouse Project

Exploring Unchartered Waters (Cohen *et al.*, 1991, p. 123)

In this problem, students are “hired” to work with Jock Crusto, a world famous oceanographer, who is exploring an uncharted ocean on the newly discovered planet of Mickey. The expedition involves three ships leaving from Port Minnie at the same time. The lead ship, Pinto, will follow a parabolic path along the curve $y = x^2$ at a speed 15 knots. The second ship, Lima, will travel in a northeasterly direction along a parabolic path such that it is always due south of the Pinto. The third ship, Verde, will also travel in a northeasterly direction along a parabolic path such that it is always due west of the Pinto. For all three ships, the vertices of their parabolas are at the origin (Port Minnie). In addition, ships must travel in such a way that the line between the Lima and the Pinto and the line between the Verde and the Pinto sweep out equal areas in equal time. The course for the Pinto and the Lima is programmed on the on-board computer, but the software is destroyed on the Verde. The task for students is to provide the x and y components for the motion of the Verde. Additional information: when the Pinto is exactly $3\sqrt{10}$ nautical miles from port, the Lima is 1.8 nautical miles south of the Pinto.

Let us model this problem in the *Geometry Expressions* by plotting three parabolas on the restricted domain, $[0, 10]$, $y = x^2$ for the Pinto, $y = ax^2$ ($a < 1$) for the Lima, and $y = bx^2$ ($b > 1$) for the Verde. Point P on the first curve represents the Pinto, point L on the

second curve represents the Lima, and point V on the third curve represents the Verde. Point M at the origin represents Port Minnie. (Fig. 7) Since all the ships left port at the same time, we introduce t as a time variable. Thus, let's constrain points P and L proportional along their corresponding curves with t as a parameter. This will also assure that the Lima is always due south of the Pinto. In order to make sure that the Verde is always due west of the Pinto, the y-coordinates of both ships have to be equal at all times. This brings to the following equation:

$$t^2 = bx^2 \rightarrow x = \frac{t}{\sqrt{b}}.$$

We will constrain point V proportional along the curve with the parameter

$$\frac{t}{\sqrt{b}}. \text{ (Fig. 7)}$$

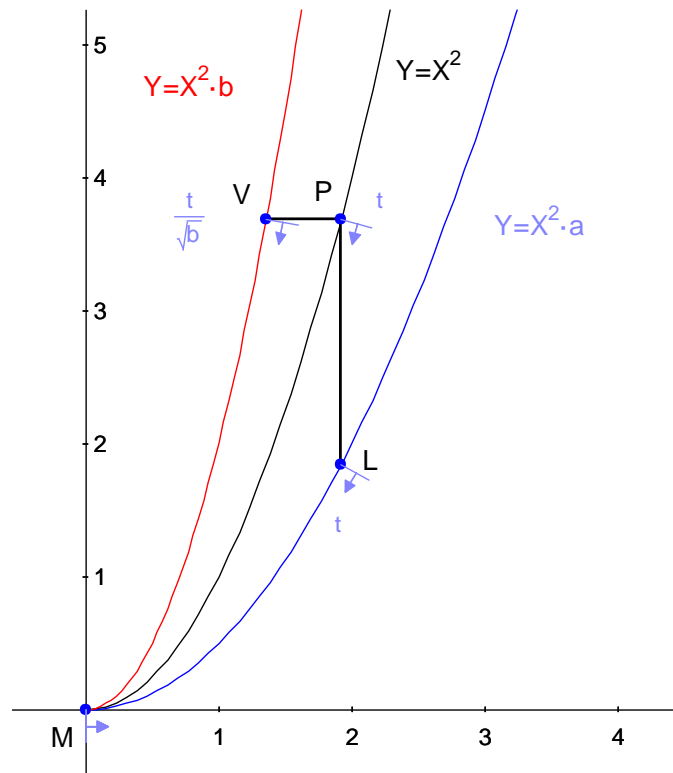


Fig. 7. Mathematical model for three ships in the Exploring Uncharted Waters problem

Thus, so far we determined the equations for the ship Verde as follows: $x(t) = \frac{t}{\sqrt{b}}$ and $y(t) = bx^2 = t^2$. In order to solve the problem, however, we need to know the value of b . The problem specifies two conditions: one that gives the distances MP and LP at a particular instance, and another one that deals with equal swept areas. The equality of areas involves both parameters, a and b , so we first need to determine the value of a using the given distances, and then find b using the equality of the swept areas.

Calculate the distances MP and LP in *Geometry Expressions*, set the distances to the given values and solve for a using TI-Nspire CAS. (Fig. 8)

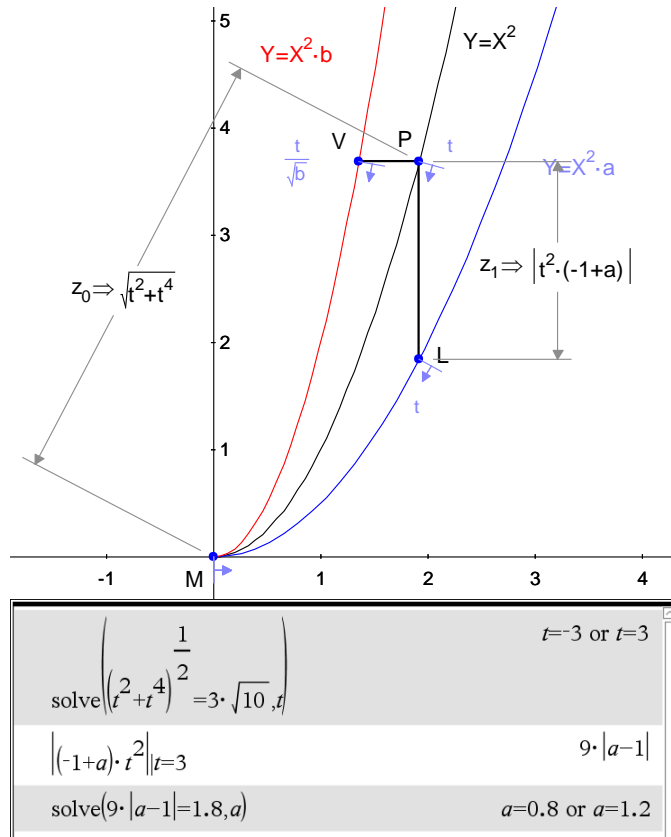


Fig. 8. Symbolic expressions for distances are generated in SGS and equations are solved in CAS

Since $a < 1$, then for the given problem we select $a = 0.8$ and lock it in the software. Note, that students can easily complete these calculations by hand, as they involve basic coordinate geometry.

Delete the expressions for distances as they are no longer needed. The next step is to determine the swept areas, which are represented by curvilinear triangles MVP and MLP . *Geometry Expressions* can calculate the areas between the curves and provide us with the symbolic expression in terms of parameters a , b , and t ; however, it will not let you put a point on the intersection of two curves. The way around this is to draw the arcs VE , PF and LG on each parabola and join each arc by line segments, EF and FG . We will then determine the curvilinear quadrilaterals, $VPFE$ and $PLGF$. (see Fig. 9a). The next step is to make points E , F , and G lie on the intersection point M . In order to do that use the point proportional constraint to specify the parametric location of each point ($x = 0$). Then we can calculate the area of each region. (see Fig. 9b)

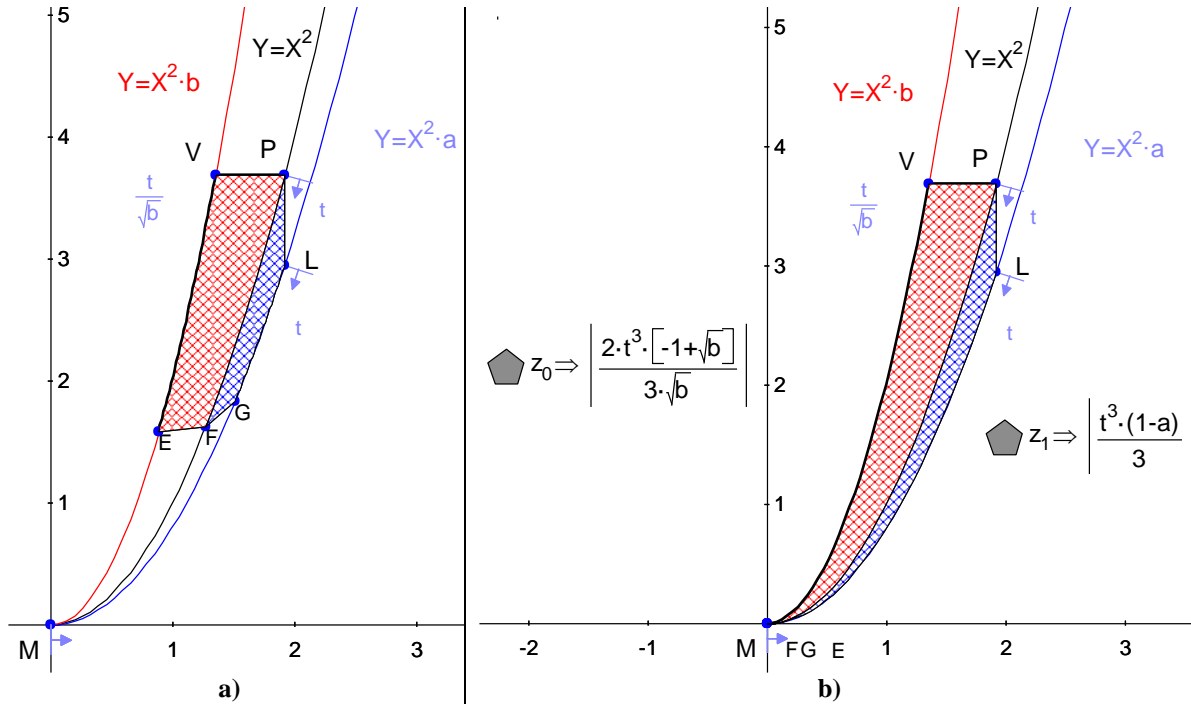


Fig. 9. Constructing curvilinear triangles and area calculations.

The last step is to set the areas equal and determine value of b . This can be easily accomplished in CAS (Fig. 10)

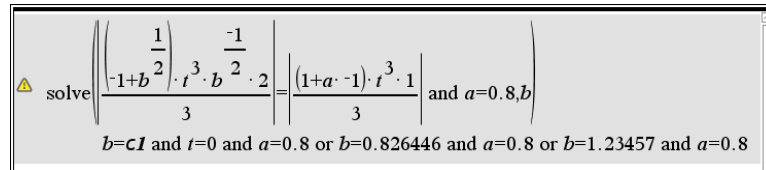


Fig. 10. Solving for b using TI-Nspire CAS software

Clearly, CAS provides all possible solutions. Since we are not interested in the $t = 0$ instance, we select the case for $b > 1$, so the answer to the problem is $b = 1.235$.

This problem also allows for a nice conceptual exploration that requires students to develop a plan for the solution and could lead to all numerical answers before students are involved in the calculus approach. The teacher can generate an applet (Fig. 11), that shows both, distances between the ships and areas swept by the corresponding segments, and allows students to drag any of the ships and vary parameters a and b to determine conditions of the problem. In this case all constructions that we completed in the software will need to be completed by the teacher. Since values of a and b require particular precision, the input for these values is left as a text box. Initial values are $a = 0.5$ and $b = 2$.

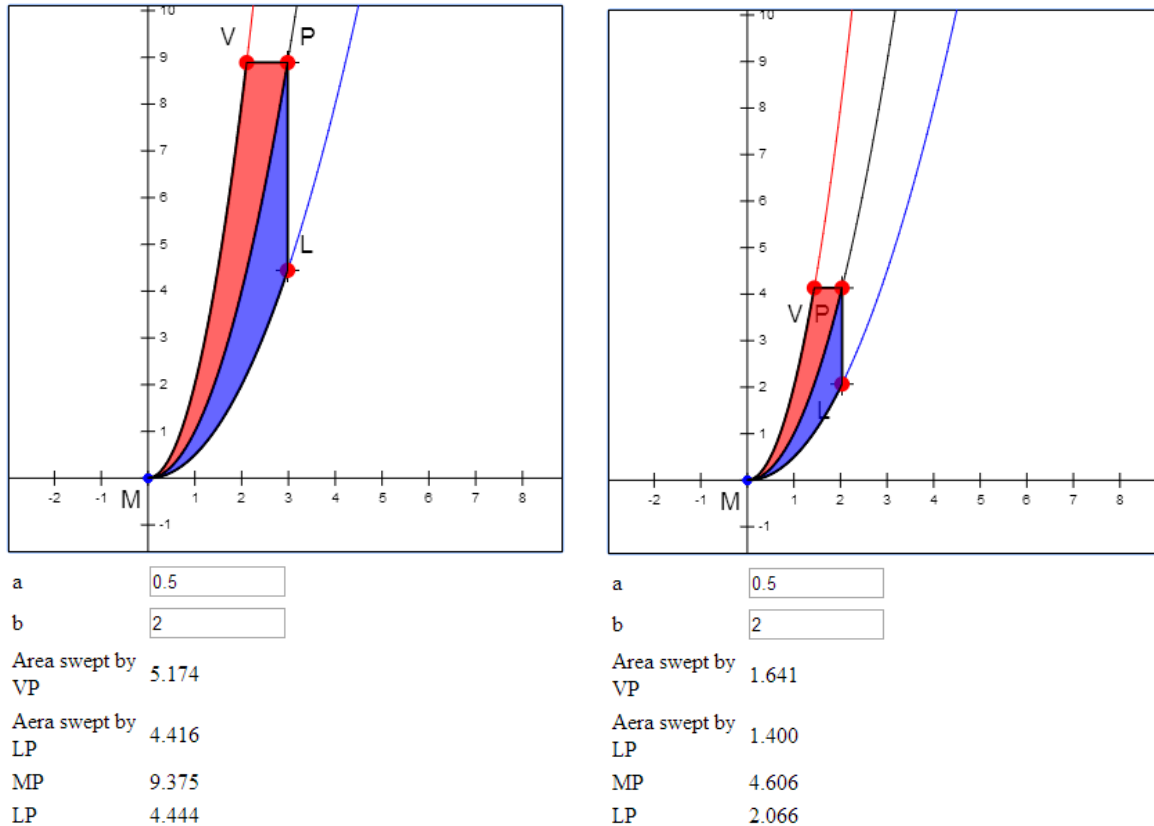


Fig. 11. HTML5/Java applet for exploration of the Exploring Uncharted Waters problem.

Random dragging of points and changes of parameters would not lead students to the answers to the problem. Students have to understand the problem and develop steps that would lead them to the solution. The first step for them is to realize that the Pinto’s motion is not affected by values of a and b . Thus, they can setup distance $MP = 3\sqrt{10} \approx 9.5$. The next step is then to vary the value of a until $LP = 1.8$. The last step is to vary b until the areas swept by two segments are the same as ships move along the curves. Thus, the applet investigation helps students develop a plan for solving problems that can then be carried out by hand.

Discussion

Let us look back to the issues raised by the mathematical community about using digital technology in mathematics education. (Risser, 2011) The first issue was whether technology should change the focus of the mathematics curriculum. The problems considered in this article are being used in different courses of calculus at high school and college level. The topics covered in these problems are part of a standard calculus curriculum. However, the technology facilitated the application of standard calculus techniques to these problems. Without technology, students often become lost in the magnitude of the mathematical expressions that need manipulating, or misled by trivial calculation errors. Symbolic geometry and CAS can raise the students above this noise level, allowing them to see each problem as a whole. The main feature of this approach is that it does not focus on “how to do calculus”, but it focuses on the conceptual understanding of major calculus ideas. It is vitally important to make sure

that students actually understand calculus and not just memorize algorithms. Going through the motions and mechanics of calculus does not ensure that students really know the concepts. This addresses the second issue: whether technology use changes how students conceptualize mathematics. (Risser, 2011) In each example, it is the student who has the responsibility of framing the problem in a meaningful way. The technology provides them with the opportunity to apply their ideas to analyze real-world problems and to model and interpret mathematical phenomena by experimenting with mathematics objects. So, back to the last issue - do the benefits of technology outweigh the costs? We believe that using technology provides learning opportunities for both teachers and students. The use of modern technology in teaching mathematics is an important part of current mathematics education. However, can we assume that students will have access to technology at home? Can we required them to purchase software for home use? It is up to the teacher or faculty to determine the most appropriate way of using technology in and out of the classroom. This article discussed various approaches to how *Geometry Expressions* can be used in teaching and learning, whether students have access to software or not. The app generation feature of *Geometry Expressions* allows teachers to provide their students with computer experimentation and exploration without the additional cost of purchasing software.

We hope that this article challenged the readers to reflect on the role that using constraint-based computer software such as *Geometry Expressions* should play in the teaching of calculus and how this software facilitates students' ability to model and solve complex problems.

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