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COMPUTER MATH SNAPHSHOTS - COLUMN EDITOR: URI WILENSKY*

Symbolic Geometry Software and Proofs

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1 Introduction

Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM] 2000) advocates a unified approach to mathematics education incorporating multiple strands in coherent focused elements. Special place in the NCTM standards is provided for a reasoning and proof strand, emphasizing its place across the mathematics curriculum:

> "Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied. In mathematically productive classroom environments, students should expect to explain and justify their conclusions." (NCTM 2000)

Dynamic Geometry software (Cabrilog 2007; Key Curriculum Press 2007) has facilitated an inductive approach to geometry. These technologies have been widely adopted in the last 20 years and a vast amount of creativity has been brought to bear on applying it to the educational process. (Kunkel et al. 2007; Scher et al. 2004; Gaulik 2002; King and Schattschneider 1997). However, this technology is not without its shortcomings. First it is

* This column will publish short (from just a few paragraphs to ten or so pages), lively and intriguing computer-related mathematics vignettes. These vignettes or snapshots should illustrate ways in which computer environments have transformed the practice of mathematics or mathematics pedagogy. They could also include puzzles or brain-teasers involving the use of computers or computational theory. Snapshots are subject to peer review. From the Column Editor Uri Wilensky, Northwestern University. e-mail: uri@northwestern.edu.

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construction-based; geometric configurations which are easy to state declaratively must be expressed in terms of sequential constructions. Whereas this action may provide an intriguing intellectual challenge, it may also be a significant distraction from the task at hand. Secondly, the existing software is numeric only, and does not have a convenient way of interacting with Computer Algebra Systems (CAS). The dynamic geometry allows students to discover results for themselves, formulate conjectures and intermediate results, examine special cases, and generate new ideas (Scher 1999; De Villiers 1999, 2006), however, it does not strongly promote the art of theorem proving.

Recently developed symbolic geometry software, such as Geometry Expressions (http://geometryexpressions.com) allows geometric and algebraic representations to coexist in the same model. Such software takes a geometric configuration and outputs algebraic expressions for quantities measured from the model. (Todd 2007). A combination allowing geometry to be modeled and expressed algebraically and then solved automatically in an algebra system provides a powerful toolkit for taking the inductive exploration-based approach facilitated by the original dynamic geometry systems to the next level, integrating geometric and algebraic exploration for developing proofs.

2 Using Geometry Expressions

Geometry Expressions differs from traditional dynamic geometry software in two ways: it is constraint based rather than construction based, and it is symbolic rather than numeric. Use of Geometry Expressions tends to exhibit the following pattern:

(1) Geometry is sketched (incidence relationships between points and lines or circles are inferred directly from the sketch)
(2) Constraints are added to the sketch: these can be qualitative, such as perpendicularity, tangency, incidence, or quantitative, such as distance, angle, slope, coordinates. Quantitative constraints may be specified numerically or symbolically.
(3) Measurements are made from the sketch. These measurements may be numeric or symbolic.

As a simple example, let's say we want to derive a formula for the altitude of a right angled triangle. The first step is to sketch a triangle ABC and a segment BD joining one vertex to the opposite side (Fig. 1a). A second step is to specify qualitative constraints, a right angle at the vertex B and one at the foot of the segment BD (Fig. 1b). A third step is to specify the lengths of the short sides of the triangle to be, symbolically, x and y (Fig. 1c). The final step is to measure the symbolic length of the altitude. The software automatically generates the expression for the length shown in Fig. 1d.

2.1 Invariance

Sometimes what is not there is more important than what is there. In mathematics, the absence of a variable in an expression points to invariance, and the invariance may have a role in proof. As an example, consider a parabola $y = x^2$. Place points at $x-h$, x, and $x + h$. Join them up to form a triangle and display its area (Fig. 2).

Notice that the area contains h, but does not contain x. Hence the area of the triangle depends on its width, but not its location. Of course, this statement relies on the symbolic geometry system to give the area of the triangle. A display of coordinates of triangle

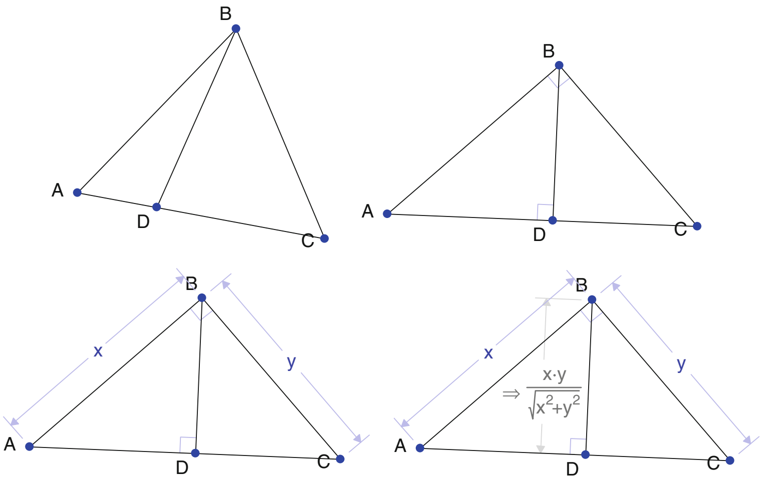


Fig. 1 Four steps in a typical usage of Geometry Expressions: **a** Sketch the geometry, D incident to AC implied from the sketch, **b** add qualitative constraints: AB perpendicular to BC and BD perpendicular AC, **c** add symbolic quantitative constraints: AB has length *x* and BC has length *y*, **d** measure the symbolic length of BD

vertices along with the median from C to AB gives a clear path to a proof of the triangle area (Fig. 3).

Comparing the *x*-coordinates of point E with points A and B students can see that point E is a midpoint of the segment AB. This statement then can be verified by using the *y*-coordinates of the three points showing that $\frac{(x-h)^2+(x+h)^2}{2} = x^2 + h^2$. Then the length of CE is easily determined as $y_E - y_C = x^2 + h^2 - x^2 = h^2$. The area of triangle ABC can be found as the sum of the areas of triangles ACE and BCE. The software will show that each triangle has area of $\frac{h^3}{2}$. The next step is to determine where this expression came from.

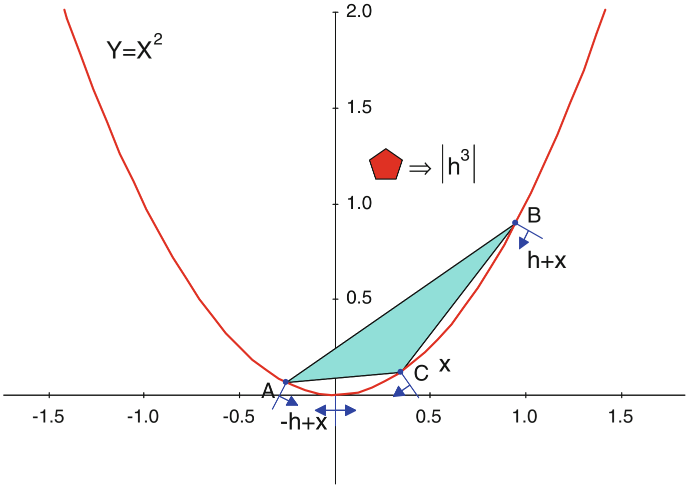


Fig. 2 Area under a chord of a parabola

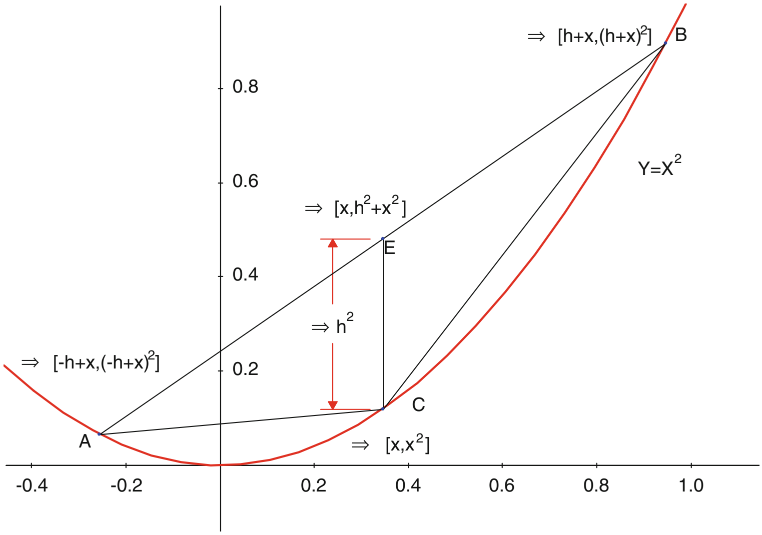


Fig. 3 E is the midpoint of AB. CE is a median of the triangle and has length h^2. CEB and CEA both have altitude h. These facts can be use to determine that the area of ABC is h^3

Assuming that CE is the base of triangle ACE we can find the altitude from A onto CE, which is given by the software as h. That is easily explained as difference in x-coordinates between points A and C (Fig. 4). The area of triangle BCE can be proved in the same way. Thus, we have proved that area of triangle ABC is h^3 and does not depend on position of point C.

The invariance of this triangle can be used, in a method which dates back to Archimedes, to derive an expression for the area under a chord of a parabola (Baki 2005).

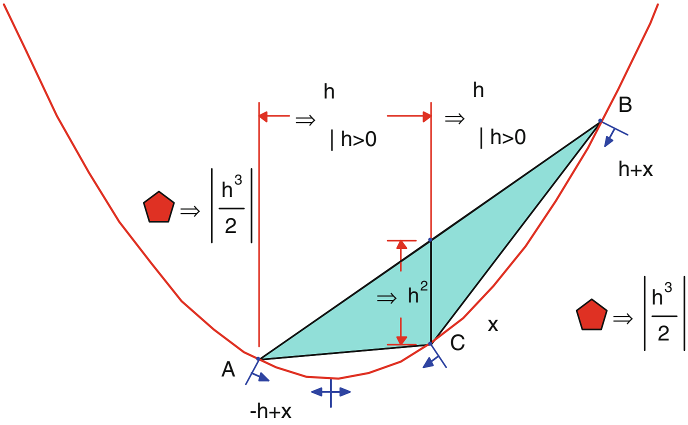


Fig. 4 Finding a path to proving area of the areas of triangles ACE and BCE. The abscissa of each point is constrained by defining its x-coordinate relative to the origin (0, 0). This constraint is indicated by the blue lines and arrows on the diagram. For example, the abscissa of the point C is defined as x, of the point B as $x + h$ and of the point A as $x-h$. The calculations assume that $h > 0$, and this assumption is displayed as Geometry Expressions calculates the horizontal distance between points A and C, and points C and B

2.2 Algebra + Geometry = Proof

Pythagorean triangles are right angled triangles with integer side lengths. If you create a specific Pythagorean triangle in Geometry Expressions, and construct its incircle, you may notice that the incircle radius is integer (Fig. 5). Can we prove that this is true in general?

With our symbolic geometry system, we can derive an expression for the radius of the incircle for a general right angled triangle (Fig. 6).

It is not immediately obvious that the radius is an integer if the side lengths are integers. However, we can rationalize the denominator as follows:

$$\frac{ab}{a+b+\sqrt{a^2+b^2}} \cdot \frac{a+b-\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}} = \frac{ab\left(a+b-\sqrt{a^2+b^2}\right)}{(a+b)^2-\sqrt{a^2+b^2}^2}$$
$$= \frac{ab\left(a+b-\sqrt{a^2+b^2}\right)}{a^2+b^2+2ab-(a^2+b^2)} = \frac{a+b-\sqrt{a^2+b^2}}{2}$$

Not quite an integer, but close. Assuming that our triangle is Pythagorean then a, b and $\sqrt{a^2+b^2}$ are all integers. Can we show that $a+b-\sqrt{a^2+b^2}$ has to be even?

We can look at three cases: a and b both even, both odd and one odd and one even.

- If a and b are both odd, $\sqrt{a^2+b^2}$ is even, hence $a+b-\sqrt{a^2+b^2}$ is even.
- If a and b are both even, then $\sqrt{a^2+b^2}$ is even, hence $a+b-\sqrt{a^2+b^2}$ is even.
- If a is even and b odd or a is odd and b even, then $\sqrt{a^2+b^2}$ is odd, hence $a+b-\sqrt{a^2+b^2}$ is even.

Having used a combination of symbolic geometry and algebra to prove our result, we can look for a more traditional geometric proof. If we specify the radius of the incircle to be r, Geometry Expressions forces us to remove one of the existing constraints in order that all constraints remain independent. Adding points incident to the circle and to the sides of the triangle (Fig. 7), we can examine the distances from these points of contact to the

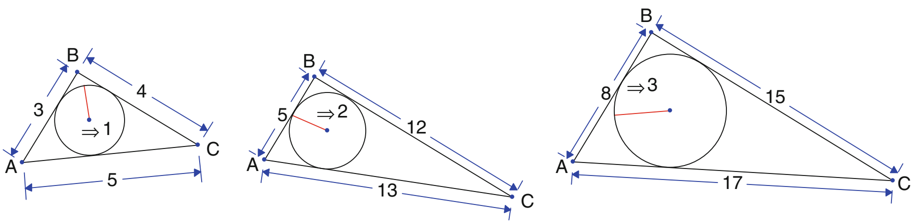
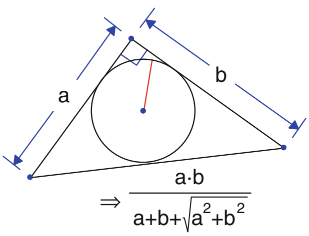


Fig. 5 Incircles of Pythagorean triangles appear to have integer radii

Fig. 6 Incircle radius for the general right angled triangle



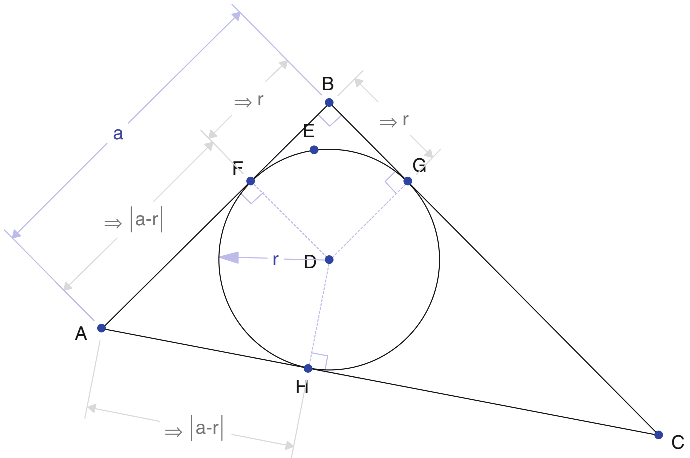


Fig. 7 Distance of points of tangency of the incircle from vertices

vertices of the triangle. Equating the sum of the two distances making up the hypotenuse with its length determined by the Pathagorean Theorem, we obtain:

$$a - r + b - r = \sqrt{a^2 + b^2},$$

which can be solved for r to yield.

$$r = \frac{a + b - \sqrt{a^2 + b^2}}{2}$$

3 Proof in Practice

In the summer of 2006, Saltire Software hosted a pair of high school student interns, under the Portland Saturday Academy's Apprenticeship in Science and Engineering program. Part of the program involved their attempting to discover mathematics for themselves using the combination of Geometry Expressions and a CAS.

Walker was going into his senior year at Lake Oswego High School. He decided to investigate the behavior of the circumcircle of a triangle in the limit as the points defining the triangle coalesced. His results on this and a variety of related problems may be viewed at (Ray 2006). His modus operandi was interesting from the point of view of this paper. He would set up specific sample problems in Geometry Expressions, derive equations which he could copy into his CAS, then take a limit in the CAS. He was then able to generalize from these specific examples to general solutions, which he would solve by hand. His final result was a collection of traditional mathematical proofs without reference to the technology, although the technology had been critical to his developing an understanding of the problem and to his development of proof paths.

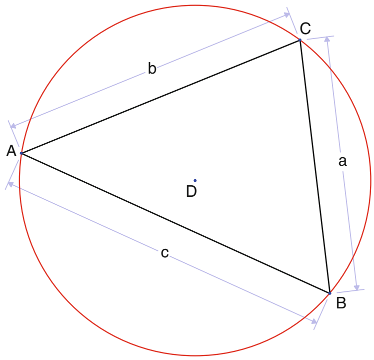
In his work, he used three different levels of confidence in a result:

(1) Numerical evidence of the form derived by direct manipulation of the geometry and examination of numerical quantities
(2) Machine proof, derived by manipulation of symbolic geometry results in a CAS

(3) Mathematical proof absent any technological aid.

Interestingly, he was not satisfied to rest at step 1 or 2, and found a result to be less than thoroughly proved if he did not accomplish step 3.

As an example of this process, we examine Walker's discovery and proof process for the limit of the circumcircle as two points coalesced. First, by dragging a numerical model, he was able to convince himself that the limit circle depended on the path along which the points coalesced. In fact he was able to formulate a hypothesis that the limit circle was tangent to the curve along which the points coalesced. His second step was to verify this result symbolically for some specific trajectories using Geometry Expressions to generate the equation of the circumcircle at a generic point on the trajectory and using Maple to take the limit. His third step was to move to pencil and paper to prove the general result. While the technology aided his thinking along the way, his final proof was a self contained piece of mathematics (Fig. 8).



We start by examining the limit of the circumradius of a triangle as two of the vertices coalesce. To define the radius of the limit circle, we examine the behavior of the circle as vertices B and C approach one another along some path.

Using the Law of Sines, we know that the circumradius R of triangle ABC with sides a, b, and c opposite the angles A, B, and C is $R = \dfrac{a}{2\sin(A)}$. When vertices B and C coalesce, side a and angle A collapse.

Let the location of B lie on the parametric curve ($x_B(t)$, $y_B(t)$), and point C lie on the parametric curve ($x_C(t)$, $y_C(t)$), and let the two points coalesce at t=0. i.e. ($x_B(0)$, $y_B(0)$)= ($x_C(0)$, $y_C(0)$)

We can define side a and angle A as the two functions a(t) and A(t), such that a(0)=0 and A(0)=0. We take the limit of R as t approaches 0:

$$\lim_{t\to 0}\left(\frac{a(t)}{2\sin(A(t))}\right) = 0/0$$

Apply l'Hopital's rule to find the limit radius:

$$\frac{a'(0)}{2\cos(A(0))A'(0)} = \frac{a'(0)}{2A'(0)}$$

If both derivatives are equal to 0, we can apply lHopital's rule repeatedly.

Fig. 8 Walker's Proof of the circumcircle limit where two points merge

4 Conclusion

A mathematician working on a proof has a fully stocked armory of tricks, of known results and of general techniques to bring to bear on a problem. One strong usage of a symbolic geometry system in teaching proof is as a conjecture creation assistant (whose conjectures happen to be correct!). The tool can help the student break a problem into tractable parts.

Traditional numerical dynamical geometry systems are useful in conjecture formulation. However, the range of conjectures which they support is limited. In such a system, equality of two measurements or direct proportion may readily be discovered, more complicated dependencies are harder to detect. By contrast, a symbolic geometry system gives an algebraic form of the dependency directly, and so arbitrarily complicated intermediate results may be formulated. For example, in our first problem, a median was added to a triangle whose area was under investigation. Displaying symbolic expressions for the length of the median along with coordinates of the triangle vertices and the foot of the median led to a proof path. This process of backward planning is characteristic of the use of a symbolic geometry system in proof. We start with the answer, then look for related results which, when taken as given yield the answer. We now can look to prove the related results, which if we are lucky, may be easier.

While symbolic geometry measurements can be used in formulating purely geometrical proofs, they can also be used as the bridge between the components of a hybrid geometric/algebraic proof. For example, when proving that the incircle of a Pythagorean triangle has integer radius, the expression for the radius is taken as given by the symbolic geometry system, and further algebraic manipulation performed to prove that it is an integer. Again the form of the symbolic expression pointed to a potential proof path. While a numeric dynamic geometry system could be used to form a conjecture in this example, it would give little help in finding a proof path.

At first glance, it might seem that a tool which creates algebra from geometry merely gives the game away. Closer experience with the tool, however, suggests that in providing a method of automatically generating intermediate results, the software can help with the strategic planning of a proof, and thus make the student more independent of teacher provided hints. Along with independence, the student may gain ownership of the problem and its solution along with motivation to push through to a fully realized mathematical proof.

References

Baki, A. (2005). Archimedes with Cabri: Visualization and experimental verification of mathematical ideas. *International Journal of Computers for Mathematical Learning, 10*, 259–270.

Cabrilog. (2007). Cabri II Plus 1.4.2 (computer software). Available at http://www.cabri.com/download-cabri-2-plus.html. Grenoble, France.

De Villiers, M. D. (1999). *Rethinking proof with the geometer's sketchpad*. Berkeley, California: Key Curriculum Press.

De Villiers, M. D. (2006). The nine-point conic: A rediscovery and proof by computer. *International Journal of Mathematical Education in Science and Technology, 37*(1), 7–14.

Key Curriculum Press. (2007). The geometer's sketchpad® dynamic geometry® software for exploring mathematics, version 4. (computer software). Available at http://keypress.com/x5521.xml. Emeryville, CA.

King, J., & Schattschneider, D. (Eds.). (1997). *Geometry turned on! Dynamic software in learning, teaching, and research*. Washington: Mathematical Association of America.

Kunkel, P., Shanan, S., & Steketee, S. (2007). *Exploring algebra 2 with the geometer's sketchpad.* Emeryville, CA: Key Curriculum Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Ray, W. (2006). Triangle circle limits. *Journal of Symbolic Geometry, 1*, 24–39. Available at www.geometryexpressions.com/journal.

Scher, D. (1999). Problem solving and proof in the age of dynamic geometry. *Micromath, 15*(1), 24–30.

Scher, D., Steketee, S., Kunkel, P., & Lyublinskaya, I. (2004). *Exploring precalculus with the geometer's sketchpad*. Emeryville, CA: Key Curriculum Press.

Todd, P. (2007). Geometry expressions: A constraint based interactive symbolic geometry system. *Lecture Notes in Computer Science, 4869*, 189–202.