

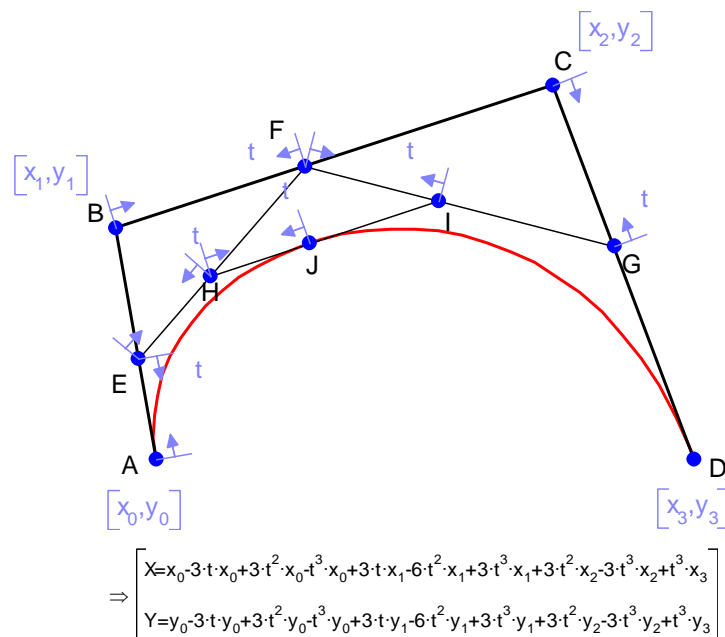
# Approximating Circular Arcs with Cubic Splines

*We use Geometry Expressions and the TI nSpire CAS to find the best cubic spline approximation to a circular arc.*

Philip Todd  
Saltire Software  
<philt@saltire.com>

## 1. Introduction

A problem came from a user of Geometry Expressions recently, which nicely illustrates the power of a symbolic geometry system used in conjunction with CAS. The user wanted to find the control points of a Bezier spline which approximated a circular arc of a given angle. Why would he want to do such a thing? Some software and some machine tools have Bezier curves as primitives, but do not have circular arcs. If you want a circle, you need to piece an approximate one together out of splines.



*Figure 1: A cubic spline constructed in Geometry Expressions. The spline's parametric equation is computed by the software*

A Bezier spline can be readily modeled in Geometry Expressions using the geometric version of de Casteljau's algorithm [ref]. We start with control polygon ABCD. These points are constrained to have coordinates  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ . Point E is constrained to be proportion  $t$  along AB. Point F is constrained to be proportion  $t$  along BC, and point G is constrained to be proportion  $t$  along CD. Points H and I are now added and constrained to be proportion  $t$  along segments EF and FG. Finally, point J is

added and constrained to be proportion t along segment HI. The spline curve is the locus of J as t runs from 0 to 1.

An important property of the spline curve is that it is tangential to the control polygon at its ends. That is at t=0, the spline curve is tangential to AB, and at t=1 it is tangential to CD.

This property can be readily verified using the TI-nSpire CAS. Geometry Expressions will automatically generate the parametric equation of the curve, and this can be copied (using Copy As / TI nSpire) into a TI-nSpire CAS worksheet.

$\left[ \begin{array}{l} x = x[0] + x[0] \cdot t \cdot -3 + x[0] \cdot t^2 \cdot 3 + x[0] \cdot t^3 \cdot -1 + x[1] \cdot t \cdot 3 + x[1] \cdot t^2 \cdot -6 + x[1] \cdot t^3 \cdot 3 + x[2] \cdot t^2 \cdot 3 + x[2] \cdot t^3 \cdot -3 + x[3] \\ y = t^3 \cdot (x[3] - 3 \cdot x[2] + 3 \cdot x[1] - x[0]) + t^2 \cdot (3 \cdot x[2] - 6 \cdot x[1] + 3 \cdot x[0]) + t \cdot (3 \cdot x[1] - 3 \cdot x[0]) + x[0] \end{array} \right]$	
$\frac{d}{dt} \left( t^3 \cdot (x[3] - 3 \cdot x[2] + 3 \cdot x[1] - x[0]) + t^2 \cdot (3 \cdot x[2] - 6 \cdot x[1] + 3 \cdot x[0]) + t \cdot (3 \cdot x[1] - 3 \cdot x[0]) + x[0] \right) \Big _{t=0}$	$3 \cdot x[1] - 3 \cdot x[0]$
$\frac{d}{dt} \left( t^3 \cdot (y[3] - 3 \cdot y[2] + 3 \cdot y[1] - y[0]) + t^2 \cdot (3 \cdot y[2] - 6 \cdot y[1] + 3 \cdot y[0]) + t \cdot (3 \cdot y[1] - 3 \cdot y[0]) + y[0] \right) \Big _{t=0}$	$3 \cdot y[1] - 3 \cdot y[0]$
$\frac{d}{dt} \left( t^3 \cdot (x[3] - 3 \cdot x[2] + 3 \cdot x[1] - x[0]) + t^2 \cdot (3 \cdot x[2] - 6 \cdot x[1] + 3 \cdot x[0]) + t \cdot (3 \cdot x[1] - 3 \cdot x[0]) + x[0] \right) \Big _{t=1}$	$3 \cdot x[3] - 3 \cdot x[2]$
$\frac{d}{dt} \left( t^3 \cdot (y[3] - 3 \cdot y[2] + 3 \cdot y[1] - y[0]) + t^2 \cdot (3 \cdot y[2] - 6 \cdot y[1] + 3 \cdot y[0]) + t \cdot (3 \cdot y[1] - 3 \cdot y[0]) + y[0] \right) \Big _{t=1}$	$3 \cdot y[3] - 3 \cdot y[2]$
5/99	

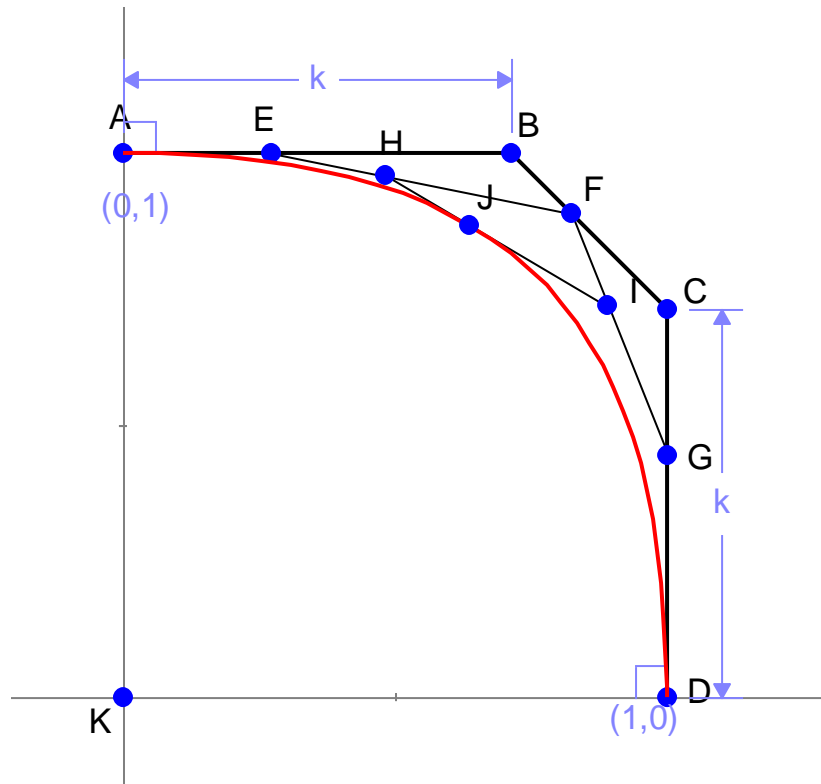
The derivatives of the x coordinate and y coordinate with respect to t evaluated at t = 0 show that the tangent aligns itself with the segment AB. Again, the derivatives evaluated at t=1 show that this tangent aligns itself with the segment CD.

## 2. Center Point on Circle

In this article, we'll use a spline to approximate a quarter circle, we'll refer to general arcs briefly in the conclusion.

We model a quarter circle of unit radius in the first quadrant. Point A is (0,1) and point D is (1,0). AB is constrained to be perpendicular to the y axis (hence the spline will be horizontal at A). CD is constrained to be perpendicular to the x-axis, and hence the spline is vertical at D.

It also seems reasonable that we should require our spline to be symmetric about x=y. We enforce this by constraining AB and BC to be the same length: k.



*Figure 2: Spline curve with control polygon constrained to guarantee tangency with the unit circle at A and D and symmetry about  $y=x$ . (proportional constraints are hidden for clarity).*

We would like to find a value for  $k$  which provides a close approximation to a quarter circle. One approach would be to require the center point of the spline curve to lie on the circle. The center point is the point with parametric location on the curve  $\frac{1}{2}$ . This can be easily created in Geometry Expressions by constraining a point's parametric location on the curve to be  $\frac{1}{2}$  (figure 3).

Geometry Expressions can supply the distance of this point from the origin. It hardly needs a CAS to find a value  $k$  such that this distance is 1. However, as we have one available, we'll use it. The resulting value of  $k$  can be copied from the TI-nSpire CAS back into geometry Expressions. Comparison with a unit circle (figure 4) shows that the fit is quite good. How good we will see soon.

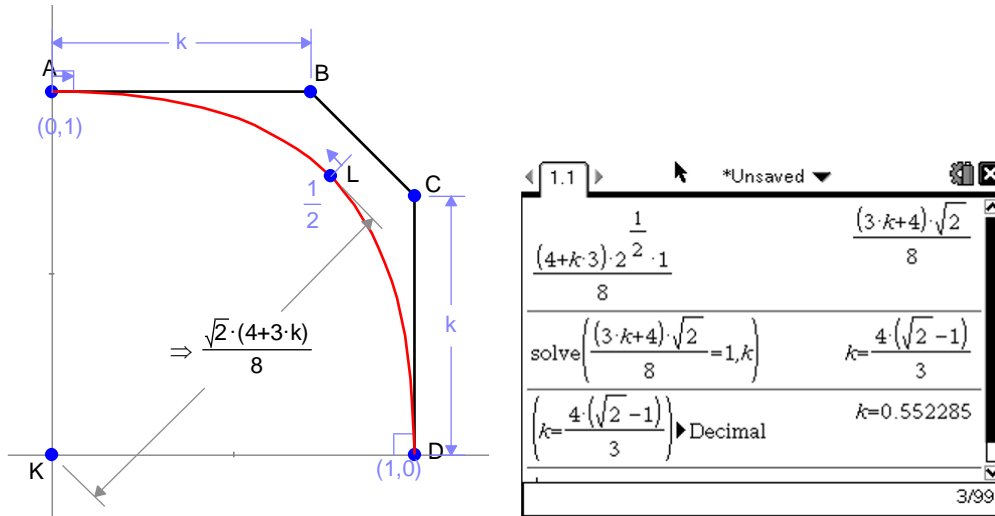


Figure 3: Distance of the center of the spline from the origin is measured in Geometry Expressions. The CAS is used to find the value of  $k$  which makes this distance 1.

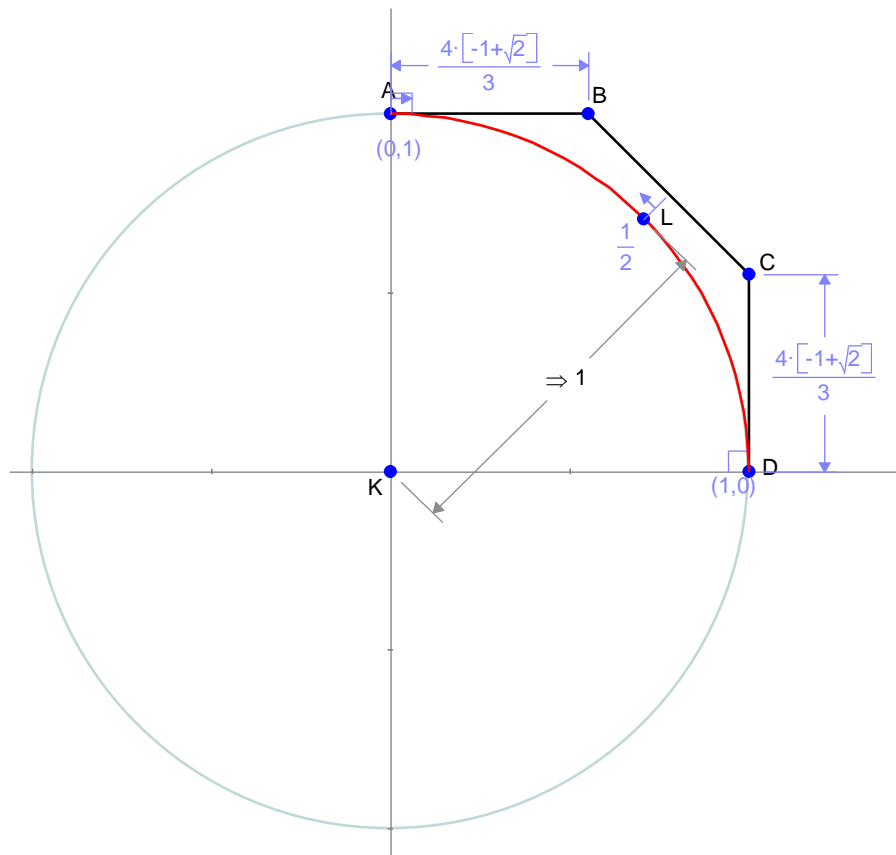
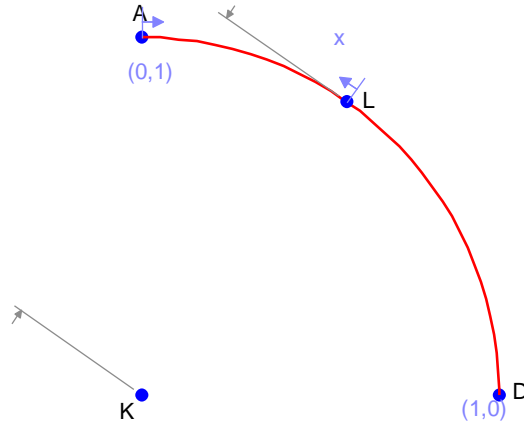


Figure 4: Values of  $k$  which make point  $L$  lie on the unit circle are copied from the CAS into Geometry Expressions allowing a visual inspection of the solution.

## 2.1 Error Analysis

To see how far away from the circle our curve lies, we change the location of L from  $\frac{1}{2}$  to a generic parametric location  $x$  (we're using  $x$  for convenience graphing in the TI nSpire, but at the risk of confusion ...  $x$  is not the  $x$  coordinate of L, but its parametric location on the curve). Geometry Expressions generates an expression for the distance of this generic point from the origin (figure 5).

$$\Rightarrow \sqrt{1+x^5 \cdot [-408+288\sqrt{2}] + x^3 \cdot [-204+144\sqrt{2}] + x^2 \cdot [34-24\sqrt{2}] + x^6 \cdot [136-96\sqrt{2}] + x^4 \cdot [442-312\sqrt{2}]}$$



$$\left( \frac{1 + \left( -408 + 2^2 \cdot 288 \right) x^5 + \left( -204 + 2^2 \cdot 144 \right) x^3 + \left( 34 + 2^2 \cdot -24 \right) x^2 + \left( 136 + 2^2 \cdot -96 \right) x^6 + \left( 442 + 2^2 \cdot -312 \right) x^4}{\sqrt{-8 \cdot (12\sqrt{2} - 17) \cdot x^6 + 24 \cdot (12\sqrt{2} - 17) \cdot x^5 - 26 \cdot (12\sqrt{2} - 17) \cdot x^4 + 12 \cdot (12\sqrt{2} - 17) \cdot x^3 - 2 \cdot (12\sqrt{2} - 17) \cdot x^2 + 1}} \right)^2$$

$$\frac{\left( -8 \cdot (12\sqrt{2} - 17) \cdot x^6 + 24 \cdot (12\sqrt{2} - 17) \cdot x^5 - 26 \cdot (12\sqrt{2} - 17) \cdot x^4 + 12 \cdot (12\sqrt{2} - 17) \cdot x^3 - 2 \cdot (12\sqrt{2} - 17) \cdot x^2 + 1 \right)^2}{-8 \cdot (12\sqrt{2} - 17) \cdot x^6 - 24 \cdot (12\sqrt{2} - 17) \cdot x^5 + 26 \cdot (12\sqrt{2} - 17) \cdot x^4 - 12 \cdot (12\sqrt{2} - 17) \cdot x^3 + 2 \cdot (12\sqrt{2} - 17) \cdot x^2 - 1}$$

$$\frac{d}{dx} \left( -8 \cdot (12\sqrt{2} - 17) \cdot x^6 - 24 \cdot (12\sqrt{2} - 17) \cdot x^5 + 26 \cdot (12\sqrt{2} - 17) \cdot x^4 - 12 \cdot (12\sqrt{2} - 17) \cdot x^3 + 2 \cdot (12\sqrt{2} - 17) \cdot x^2 - 1 \right)$$

$$-4 \cdot (12\sqrt{2} - 17) \cdot x \cdot (12x^4 - 30x^3 + 26x^2 - 9x + 1)$$

$$\text{solve}(-4 \cdot (12\sqrt{2} - 17) \cdot x \cdot (12x^4 - 30x^3 + 26x^2 - 9x + 1) = 0, x) \quad x=0 \text{ or } x = \frac{-(\sqrt{3}-3)}{6} \text{ or } x = \frac{1}{2} \text{ or } x = \frac{\sqrt{3}+3}{6} \text{ or } x=1$$

Figure 5: The distance from the origin of point at parametric location  $x$  on the spline curve is copied from Geometry Expressions into the TI-nSpire CAS. The expression is squared, differentiated and solved.

To find the values of  $x$  for which this distance is maximal or minimal, we first the expression into our CAS. Optimizing the distance is the same as optimizing its square, so our first step inside the CAS is to square the expression, thus getting rid of the square

root, and making it more likely that the derivative will fit on the page. Next, differentiate, then solve (figure 5).

We observe that we have optima at 0, 1/2 (guaranteed by symmetry), 1 and at two other values. By construction, the curve lies on the unit circle at 0, 1/2 and 1, leaving two points of interest, which are symmetric about  $x = 1/2$  and should, by the symmetry of the figure, lie at the same distance from the origin. This then is the quantity we are interested in. We simply copy it from the CAS and paste into the Geometry Expressions, replacing  $x$  as the parametric location of point L (figure 6).

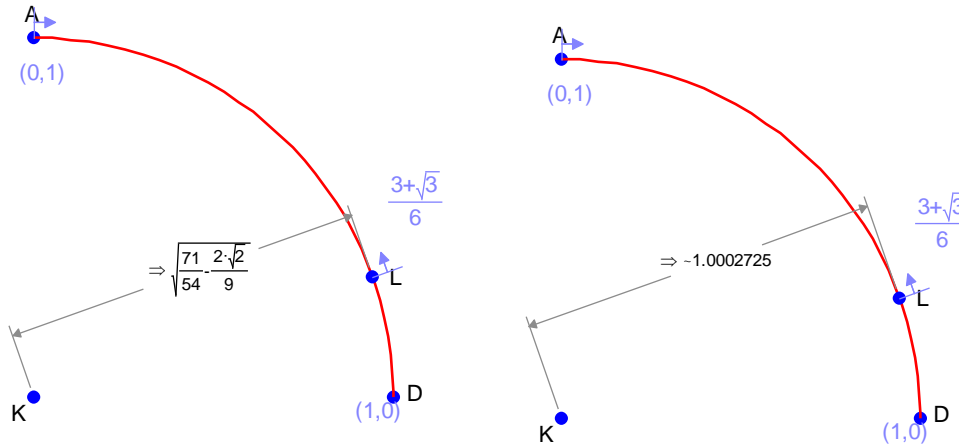


Figure 6: Maximal distance of point L is obtained by pasting the solution for  $x$  into the parametric location constraint for point L. Numeric display shows the error to be less than  $3/10,000$

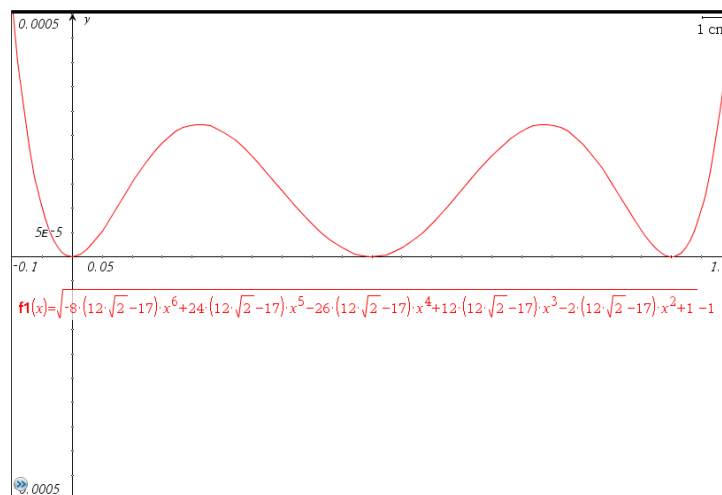


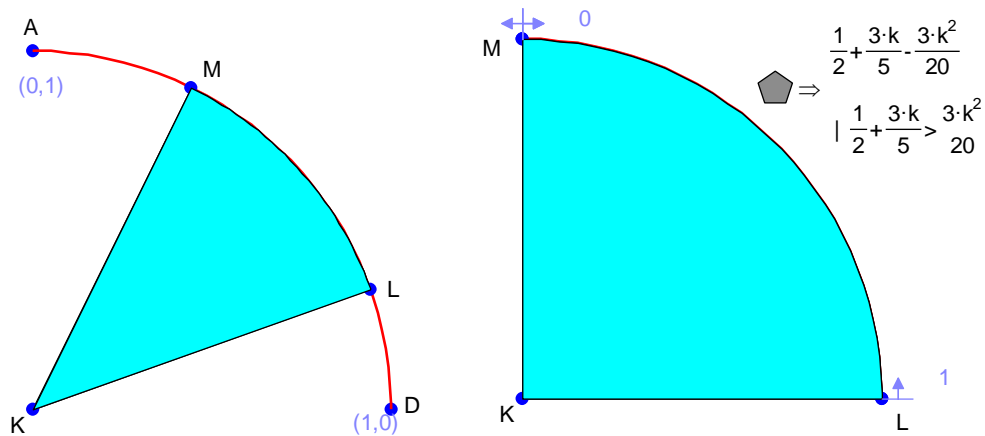
Figure 7: Error graphed as a function of parametric location on the curve.

An error function may be obtained by subtracting 1 (the radius of the circle) from the distance function measured in figure 5. This is displayed in figure 7. We observe that this spline curve always errs on the side of being too large. For some applications this

may be a benefit, however, we might ask ourselves whether it would be possible to improve our estimate if we allowed the spline to err both inside and outside the circle. We'll look at one such curve in the next section.

### 3. Equal Area

One way of ensuring that the spline errs both on the inside and on the outside of the unit circle is to insist that the area under the spline is the same as the area under the quarter circle. To measure the area under the spline in Geometry Expressions, you need to create first an arc LM on the spline, then draw line segments LK and KM (where K is the origin). You can then select the segments and the arc and create a polygon. As a final step, you will make sure L and M lie at the start and end of the curve by constraining their parametric locations to 0 and 1.



$$\frac{1}{2} + \frac{3 \cdot k}{5} - \frac{3 \cdot k^2}{20}$$

$$\left| \frac{1}{2} + \frac{3 \cdot k}{5} - \frac{3 \cdot k^2}{20} \right| > \frac{\pi}{20}$$

$$\left[ \frac{1}{2} + \frac{3 \cdot k}{5} + k^2 \cdot \frac{-3}{20} \cdot \frac{1}{2} + \frac{3 \cdot k}{5} + k^2 \cdot \frac{-3}{20} > 0 \right]$$

$$\left[ \frac{-3 \cdot k^2}{20} + \frac{3 \cdot k}{5} + \frac{1}{2} \cdot \frac{-3 \cdot k^2}{20} + \frac{3 \cdot k}{5} + \frac{1}{2} > 0 \right]$$


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$$\text{solve} \left( \frac{1}{2} + \frac{3 \cdot k}{5} + k^2 \cdot \frac{-3}{20} = \frac{\pi}{4}, k \right)$$

$$k = \frac{-\sqrt{-3 \cdot (5 \cdot \pi - 22)} - 6}{3} \text{ or } k = \frac{\sqrt{-3 \cdot (5 \cdot \pi - 22)} + 6}{3}$$


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$$\left( k = \frac{-\sqrt{-3 \cdot (5 \cdot \pi - 22)} - 6}{3} \text{ or } k = \frac{\sqrt{-3 \cdot (5 \cdot \pi - 22)} + 6}{3} \right) \blacktriangleright \text{Decimal}$$

$$k = 0.551778 \text{ or } k = 3.44822$$

Figure 8: The area under the spline is measured in Geometry Expressions by creating an arc LM on the spline, connecting L and M to the origin and creating a curve-sided polygon KLM. Finally, L and M are constrained to lie at parametric locations 1 and 0 on the curve. In our CAS, we can find the value of k which makes the area under the spline equal to the area under a quarter unit circle.

Copying the area value into the CAS we can equate it with  $\pi/4$  (the area under the quarter unit circle), and solve for k. We have two solutions. The first is numerically close to the value of k which puts the center point on the unit circle. The second is too far removed to be reasonable. Figure 9 shows what they look like:

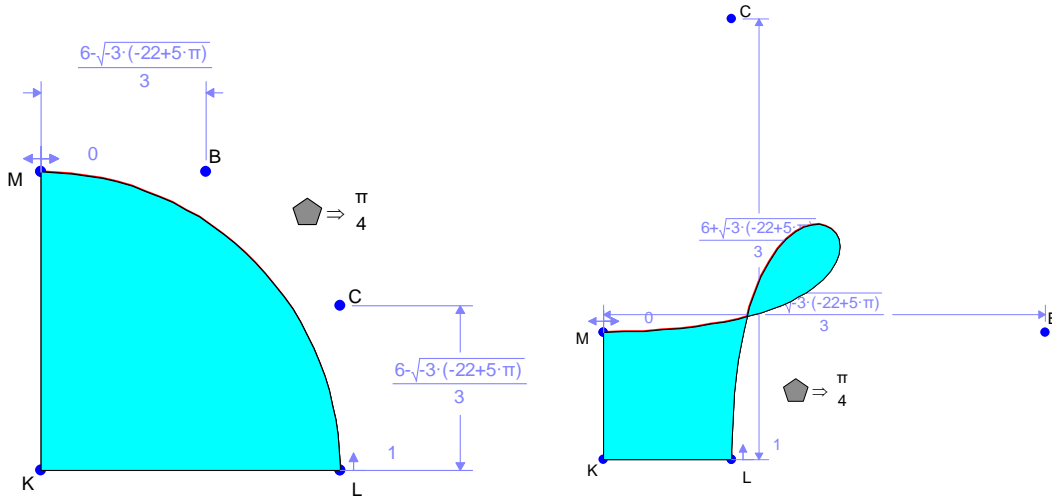


Figure 9: two solutions yielding the same area under the spline. Only one is a good approximation for the quarter circle!

We can paste this value back into Geometry Expressions for k, and let the software derive the distance from the origin to the point on the curve at parametric location x (figure 19). This can in turn be copied into TI-nSpire for graphing (figure 10).

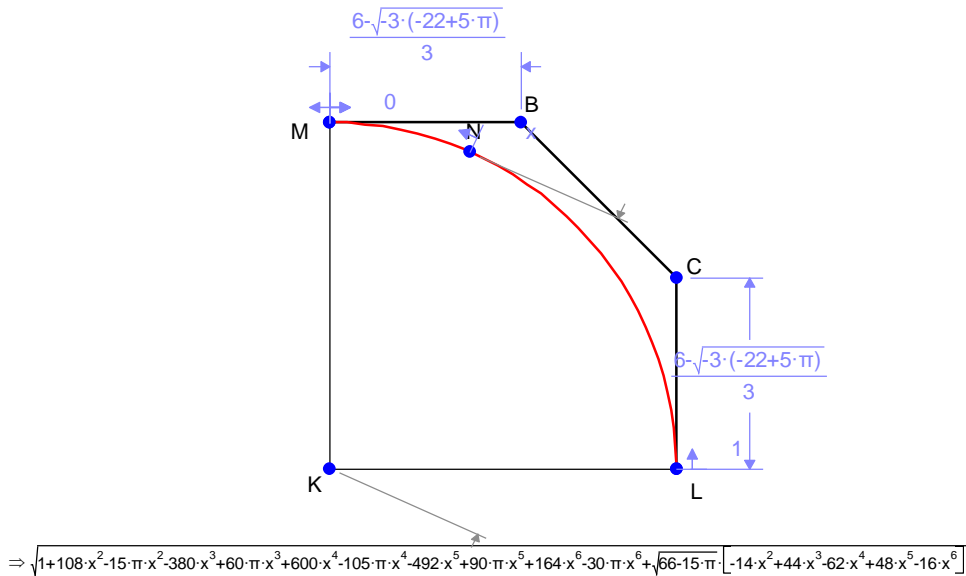


Figure 10: distance to the origin for point at parametric location x on the equiareal approximation



Analysis of the graph shows that while we have succeeded in reducing the extent to which the approximate curve overshoots the circle, we have done so at a cost of undershooting it by almost as much at the center point.

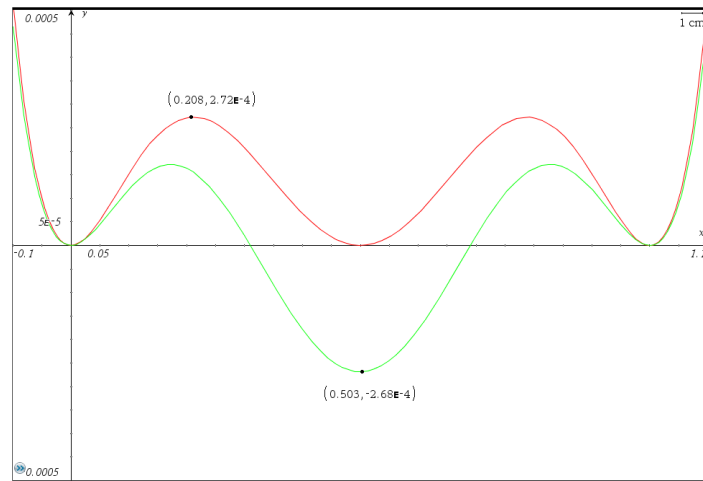


Figure 11: Radius for the equiareal approximation is copied into TI-nSpire from Geometry Expressions, and plotted next to the area for the central-point approximation.

However, this suggests an approach for finding a better solution. Can we find a value of  $k$  such that the error function undershoots by the same amount that it overshoots?

#### 4. Minimizing the maximum error

Using Geometry Expressions, we can generate a distance function for a point at parametric location  $x$ , copy this into TI-nSpire CAS and differentiate to find the maxima and minima (figure 12).

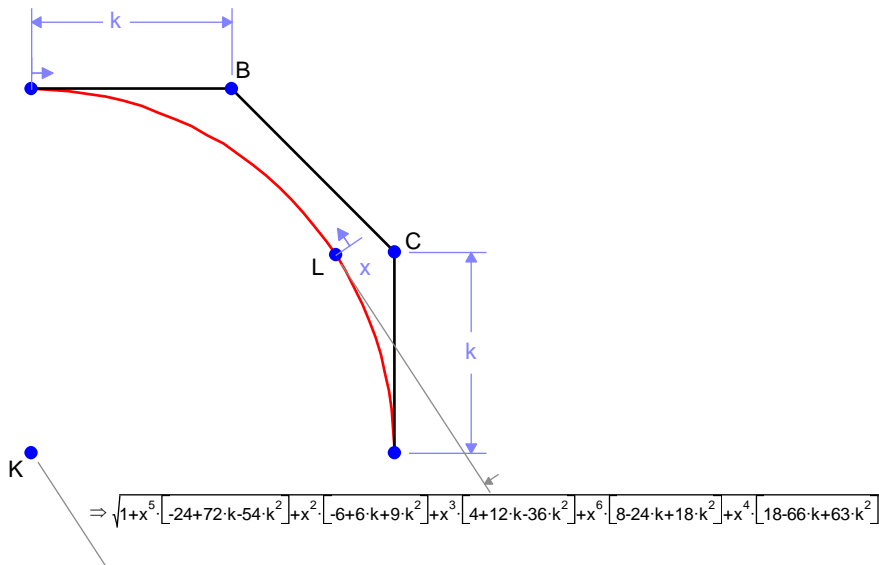


Figure 12: Distance from the origin of the generic point on the spline with parameter  $k$

$$\frac{(1+(-24+k-72+k^2)\cdot 54)\cdot x^5+(-6+k-6+k^2\cdot 9)\cdot x^2+(4+k-12+k^2\cdot 36)\cdot x^3+(8+k-24+k^2\cdot 18)\cdot x^6+(18+k-66+k^2\cdot 63)\cdot x^4)^2}{\sqrt{2\cdot(9\cdot k^2-12\cdot k+4)\cdot x^6-6\cdot(9\cdot k^2-12\cdot k+4)\cdot x^5+3\cdot(21\cdot k^2-22\cdot k+6)\cdot x^4-4\cdot(9\cdot k^2-3\cdot k-1)\cdot x^3+3\cdot(3\cdot k^2+2\cdot k-2)\cdot x^2+1}}$$

$$\frac{(\sqrt{2\cdot(9\cdot k^2-12\cdot k+4)\cdot x^6-6\cdot(9\cdot k^2-12\cdot k+4)\cdot x^5+3\cdot(21\cdot k^2-22\cdot k+6)\cdot x^4-4\cdot(9\cdot k^2-3\cdot k-1)\cdot x^3+3\cdot(3\cdot k^2+2\cdot k-2)\cdot x^2+1})^2}{2\cdot(9\cdot k^2-12\cdot k+4)\cdot x^6-6\cdot(9\cdot k^2-12\cdot k+4)\cdot x^5+3\cdot(21\cdot k^2-22\cdot k+6)\cdot x^4-4\cdot(9\cdot k^2-3\cdot k-1)\cdot x^3+3\cdot(3\cdot k^2+2\cdot k-2)\cdot x^2+1}}$$

$$\frac{d}{dx}(2\cdot(9\cdot k^2-12\cdot k+4)\cdot x^6-6\cdot(9\cdot k^2-12\cdot k+4)\cdot x^5+3\cdot(21\cdot k^2-22\cdot k+6)\cdot x^4-4\cdot(9\cdot k^2-3\cdot k-1)\cdot x^3+3\cdot(3\cdot k^2+2\cdot k-2)\cdot x^2+1)$$

$$12\cdot(9\cdot k^2-12\cdot k+4)\cdot x^5-30\cdot(9\cdot k^2-12\cdot k+4)\cdot x^4+12\cdot(21\cdot k^2-22\cdot k+6)\cdot x^3-12\cdot(9\cdot k^2-3\cdot k-1)\cdot x^2+6\cdot(3\cdot k^2+2\cdot k-2)\cdot x$$

solve( $12\cdot(9\cdot k^2-12\cdot k+4)\cdot x^5-30\cdot(9\cdot k^2-12\cdot k+4)\cdot x^4+12\cdot(21\cdot k^2-22\cdot k+6)\cdot x^3-12\cdot(9\cdot k^2-3\cdot k-1)\cdot x^2+6\cdot(3\cdot k^2+2\cdot k-2)\cdot x=0$ , $x$ )

$$x = \frac{-\sqrt{3\cdot k^2-20\cdot k+12}\cdot|3\cdot k-2|-9\cdot k^2+12\cdot k-4}{2\cdot(9\cdot k^2-12\cdot k+4)} \text{ or } x = \frac{\sqrt{-3\cdot k^2-20\cdot k+12}\cdot|3\cdot k-2|+9\cdot k^2-12\cdot k+4}{2\cdot(9\cdot k^2-12\cdot k+4)} \text{ or } x=0 \text{ or } x=\frac{1}{2} \text{ or } x=1$$

Figure 13: extrema for the curve of figure 12 calculated in the TI-nSpire CAS

The extrema lie at  $x = \frac{1}{2}$  and at two more complicated points. We can copy and paste the complicated points into Geometry Expressions and let it worry about the algebra. The distance value (figure 13) is surprisingly simple.

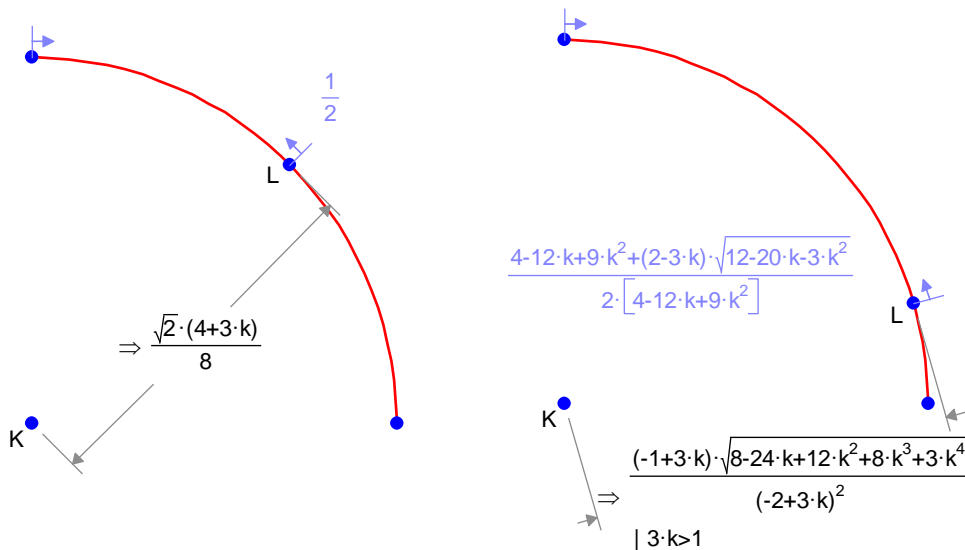


Figure 13: Extreme values for the distance from the origin on the spline curve with parameter  $k$

In the TI-nSpire CAS we define a variable error1 to be the distance at the midpoint subtracted from 1. We define error2 to be the distance at the more complicated extremum minus 1. The CAS will find, numerically, a value of k which equates these errors (figure 14).

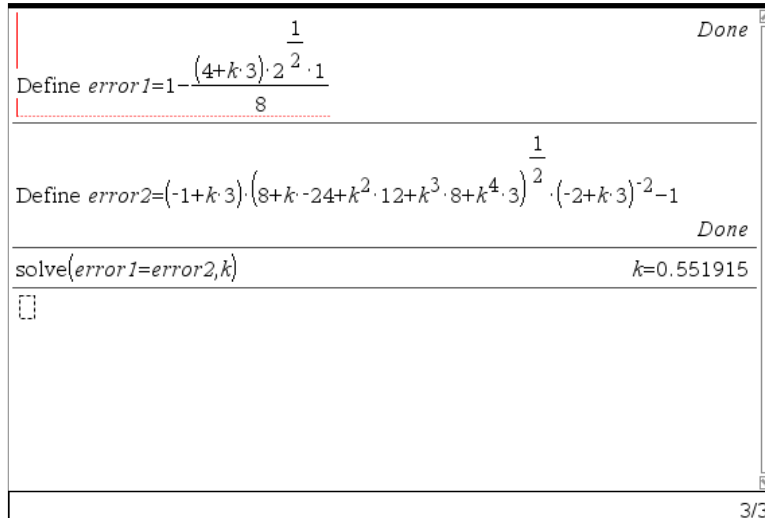


Figure 14: Numerical solution for k, equating the absolute values of the two errors

A plot of the error functions (figure 15) reveals that the minimax solution reduces the maximal error by over 25% compared with either of the other approximations.

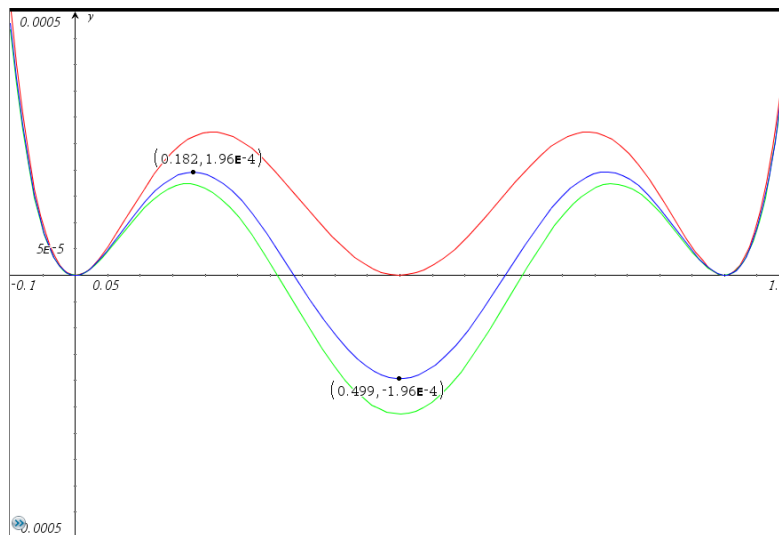


Figure 15: Error curve for the minimax solution lies between the error curves for the central point solution and the equiareal solution.

## 5. Conclusion

We have looked at modeling a quarter circle. It is also useful to be able to approximate a general portion of a circular arc. The development of spline approximations for general arcs yields more complicated formulas, but you can use a similar process, and the CAS

can take care of the details. You do need to be careful with the minimax solution as the location of the extremum is complicated enough that computing its value can cause problems both for Geometry Expressions and the TI nSpire.

A straightforward, but interesting analysis would be to investigate the improvement in accuracy to be obtained by approximating a quarter circle by two 45-degree spline approximations.

Another question would be this: let's say we are approximating a full circle by splines, each of which approximates one quarter of the circle. It is a little artificial to insist that the spline meets the circle at the axes. Why not allow the spline to start and finish at a radius slightly below that of the circle being approximated? In this way we could essentially translate the error curve for the center point approximation vertically down by half its amplitude and achieve a 50% reduction in error.

The huge benefit of having such powerful software tools as Geometry Expressions and the TI nSpire CAS working together under your direction is that they allow you to explore many interesting variants of a problem in the time it might take to investigate a single variant by hand.