# Looking Forward to Interactive Symbolic Geometry 

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Interactive geometry software has features for facilitating discovery based learning. It has been widely adopted in the last twenty years and a vast amount of creativity has been brought to bear on applying it to the educational process [1,2]. This is not to say that it is without its limitations.

Interactive geometry systems are constructive rather than constraint based. In a construction based system, for example, to create an incircle of a triangle, one constructs first the intersection of the angle bisectors, then the perpendicular from one side through this point, then the circle centered at the intersection point whose circumference goes through the foot of the perpendicular. In a constraint based system, by contrast, one sketches a circle inside the triangle, then constrains the circle to be tangent to each side of the triangle. A construction based system models the process of geometry as practiced a couple of millennia ago by the ancient Greeks, whereas a constraint based system models the process of geometry practiced nowadays in Computer Aided Design systems. Further, interactive geometry programs are purely numeric and thus are limited in their ability to reinforce the important connections between geometry and algebra. In this paper, we briefly describe the features of an interactive symbolic geometry system which addresses both these limitations and provides an interesting complement to existing DGS programs.

## Geometry in a classroom of the future

Constraint based symbolic geometry software should make an impact on mathematics education in the next 20 years. At the time of writing, although symbolic geometry software exists [3] it has not been deployed long enough for a significant body of knowledge to have amassed on how to use it in a classroom setting. With the reader's indulgence, then, we are going to embark on a little time travel into a high school of the future, let's say 2016. (We note again that although our imaginary high school is in the future, the software illustrated exists today. Specifically, the models described in this article were created using the symbolic geometry program Geometry Expressions [3]. We further assume that the students of the future will have access to Computer Algebra Systems (CAS) in much the same way that the students of today have access to graphing calculators. We have used Maple [4] as our CAS.)

Our first stop will be a sophomore class. The students all have computers: some are laptops, some tablets, some look like calculators. All have symbolic geometry software and CAS. They are investigating relationships between the radii of incircles and excircles of a triangle. As a warm up, they create a formula for the area of a triangle. The students first sketch a triangle $A B C$, then add length constraints to each side, specifying that $A B$ is length $c, B C$ is length $a$, and $A C$ is length $b$. They select the triangle and create a polygon, then request the software to display the area of the polygon. Their symbolic geometry program automatically creates the expression for the area shown in Fig 1. (The shaded red pentagon is an icon representing the fact that the quantity displayed is an area).


Figure 1: "Geometry Expressions" computes a formula for the area of a triangle given the length of three sides

Ms. Johnson informs her class that this beautiful formula is due to Heron of Alexandria, and frequently appears in textbooks in the form:

$$
\begin{equation*}
\sqrt{s(s-a)(s-b)(s-c)} \text { where } s=\frac{a+b+c}{2} \tag{1}
\end{equation*}
$$

She asks the class to verify that (1) is indeed equivalent to the equation generated by their symbolic geometry system.

They have Heron's formula now in two different formats, but how could they prove it? Ms. Johnson prompts her class to constrain a triangle in terms of the altitude and the two non base sides (fig. 2), and then to show the formula for the length of the base.


Figure 2:Length of the base in terms of the other sides of the triangle and its altitude

The class can easily see how they could have derived this expression using Pythagoras' Theorem. They agree that if they equate this quantity with b , then solve for x they will have a formula for the altitude in terms of the side lengths. Strategically, they have finished the problem. The only issue is the matter of solving the equation. They entrust this to an algebra system

$$
\begin{aligned}
& \text { >> solve (sqrt ( } \left.\left.a^{\wedge} 2-x^{\wedge} 2\right)+ \text { sqrt }\left(c^{\wedge} 2-x^{\wedge} 2\right)=\mathrm{b}, \mathrm{x}\right) ; \\
& \qquad \begin{array}{r}
\frac{\sqrt{2 a^{2} b^{2}-a^{4}-c^{4}+2 c^{2} a^{2}+2 c^{2} b^{2}-b^{4}}}{2 b}, \\
\\
-\frac{\sqrt{2 a^{2} b^{2}-a^{4}-c^{4}+2 c^{2} a^{2}+2 c^{2} b^{2}-b^{4}}}{2 b}
\end{array}
\end{aligned}
$$

Two solutions, but the second is negative. The expression under the square root is in expanded form so it is difficult to compare with Heron. They use the CAS's factor() command to straighten this out:

```
> factor(%);
```

$$
\frac{\sqrt{-(b+a-c)(b+a+c)(-b+a+c)(-b+a-c)}}{2 b}
$$

With this equation for the altitude, the students immediately multiply by half the base and verify Heron's formula.

The class's next stop is the incircle, what is its radius? Again their symbolic geometry system automatically gives them a formula in terms of the lengths of the sides of the triangle (fig 3).


Figure 3:Automatically generated expression for the radius of the incircle
Some discussion follows, one student notices the formula is symmetric in $\mathrm{a}, \mathrm{b}$ and c . Another points out it is quite similar to Heron's formula. That's exactly the cue Ms Johnson has been waiting for: she challenges them to work out how to change the area formula to get the radius. Some get to work in their CAS, copying the two expressions from the symbolic geometry system, Others work directly in the geometry system (figure 4). Ms Johnson strolls through the aisle and gently prompts:
"The area formula has a 4 in the denominator, and the radius has a 2 , how would you change the 4 to a 2 ? Perhaps multiply by 2 ?"
"Now the area has a $\sqrt{a+b+c}$ in the numerator which is not there in the radius, how would you get rid of it from the numerator - that's right divide by it. But now the radius has this term in the denominator, so you'd need to divide by it again."
"So you've divided by $\sqrt{a+b+c}$ twice. What is that equivalent to? That's right, dividing by $(\mathrm{a}+\mathrm{b}+\mathrm{c})$. ."


Figure 4: Relation of the incircle radius to the triangle area
They identify in this way that the expression for the radius of the incircle is twice the area of the triangle divided by the sum of the sides.

How could this be proved, their teacher asks. She suggests creating the picture of figure 5 where the triangle is divided into three parts, and the radius of the incircle is specified.


Figure 5:Area decomposition proof of the incircle radius
The class has a collective aha moment, the radius of the incircle is the altitude of each of the smaller triangles. Ms. Johnson suggests getting a formula for the area of the large triangle in terms of the radius by adding the smaller triangles, and the class quickly obtains:

$$
\begin{equation*}
A=\frac{a r}{2}+\frac{b r}{2}+\frac{c r}{2} \tag{2}
\end{equation*}
$$

Some in the class type this directly into their CAS to solve for r , but Ms. Johnson gently scolds them: this is too simple an equation to use your CAS - solve it by hand please.

Ms. Johnson ends the session by defining the excircle $[5,6]$ as a circle which, like the incircle, is tangent to all three sides of the circle, but whereas the incircle lies inside the circle, the excircle lies outside the circle. A quick illustration (figure 6.) clears up any confusion. She asks the class to find the radius of one of the excircles. They do this quickly in their symbolic geometry systems. As the class departs, Ms. Johnson suggests they might want to think about how they could prove this, but we suspect they are more likely thinking about lunch.


Figure 6: A triangle has 3 excircles, tangent to its sides but external to the triangle
When the class reconvenes the next day, their teacher reminds them of their result for the radius of the incircle. She splits them into three groups, one for each excircle, and asks them to derive an expression for their excircle in terms of the triangle area. She instructs the groups not to divulge their radius. She then asks the students not only to write down the radius of their own incircle, but also to guess the radii of the other groups' circles.

The team investigating the excircle external to BC presents their results (fig. 7)


Figure 7: Relation between the excircle radius and the triangle area
Their guesses for the other two excircles are accurate:

$$
\begin{equation*}
\frac{2 A}{a-b+c}, \frac{2 A}{-a+b+c} \tag{3}
\end{equation*}
$$

Ms. Johnson reminds the class how they proved the incircle radius by dividing the triangle into three smaller triangles whose common point is the center of the incircle. She asks them for suggestions about using a similar strategy with the excircle. Despite some initial confusion as to how to divide a triangle using an external point, with a little guidance from Ms. Johnson, the class comes up with the pictures in Fig 8


Figure 8:Area decomposition proof of the excircle radius

The connection between the geometry and the algebra immediately becomes apparent, the area of the original triangle can be found by adding two of the smaller triangles and subtracting the third.

$$
\begin{equation*}
A=-\frac{a r}{2}+\frac{b r}{2}+\frac{c r}{2} \tag{4}
\end{equation*}
$$

This corresponds to the term in the denominator of the radius expression being the sum of two lengths minus the third. This time nobody reaches for their CAS, solving for $r$ by hand. Ms. Johnson writes the results of the class investigation on the board, a set of formulas for the radii of the incircle and the three excircles.

$$
\begin{align*}
& r_{0}=\frac{2 A}{a+b+c} \\
& r_{1}=\frac{2 A}{a+b-c}  \tag{5}\\
& r_{2}=\frac{2 A}{a-b+c} \\
& r_{3}=\frac{2 A}{-a+b+c}
\end{align*}
$$

She assigns the class homework: to come up with expressions involving radii and area, but without the lengths. In other words, she says, are there any relationships between the radii and area which are true for all triangles?

Unfortunately we don't get to visit Ms. Johnson's class to see what they came up with we are off to Mr. Ford's pre-calculus class tomorrow. Would anyone recognize that multiplying all the radii would give a denominator closely related to the square of the area?

$$
\begin{equation*}
r_{0} r_{1} r_{2} r_{3}=\frac{16 A^{4}}{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)} \tag{6}
\end{equation*}
$$

And hence:

$$
\begin{equation*}
r_{0} r_{1} r_{2} r_{3}=\frac{A^{4}}{A^{2}}=A^{2} \tag{7}
\end{equation*}
$$

Would someone identify that inverting the radius expressions would make them amenable to addition? Would anyone try adding the reciprocals of the excircle radii?

$$
\begin{equation*}
\frac{1}{r_{1}}+\frac{1}{r_{2}}+\frac{1}{r_{3}}=\frac{a+b-c}{2 A}+\frac{a-b+c}{2 A}+\frac{-a+b+c}{2 A}=\frac{a+b+c}{2 A}=\frac{1}{r_{0}} \tag{8}
\end{equation*}
$$

Perhaps not, but I'm sure Ms. Johnson will have a set of prompts to elicit these results from the class when they next meet.

Mr. Ford meanwhile, has posed his class the problem of moving a ladder round a corner. What, he asks, is the longest ladder that you can carry round a right angled junction between a corridor of width x and a second corridor of width y ? [6]

He works with them to set up the problem up in their symbolic geometry system. They create a pair of axes which will be one pair of walls. They offset the other walls of the corridor by x and y , and constrain the length of the ladder to be L. Dragging the foot of the ladder along the wall lets the class experiment with different ladder lengths (fig. 9)


Figure 9: Moving a ladder round a corner
Mr. Ford assigns different values of x and y to different students and asks them to find by trial and error a value for L which only just clears the corner. He writes the values in a table on his whiteboard, but the group is unable to guess a general form from the numbers. He asks the class if there is a way to simplify the problem. He elicits the answer he is clearly looking for: what if we make the two corridors the same width?

They decide that if the ladder only just fits, then its center will hit the corner of the wall. Mr. Ford suggests they create the locus of the center of the ladder. Asked what kind of curve this is, the class suspects it may be a quarter circle, or part of an ellipse. The symbolic geometry system shows the equation of the locus, which is quickly identified as a circle with radius $\mathrm{L} / 2$ (fig. 10).


Figure 10: Locus of the midpoint of the ladder is a quarter circle
What, asks Mr. Ford, is the distance of D from the origin? The class immediately generates the result with their geometry system, but Mr. Ford asks how they could have worked this out without the computer. With a little prompting, the class agrees that this is the diagonal of a square of side $x$, and is an easy result of Pythagoras' Theorem.

What then, is the length of the largest ladder that will fit round the corner? Several of the students have already put the answer into their geometry system to confirm it by the time he gets the official response:

$$
\begin{equation*}
L=2 \sqrt{2} x \tag{9}
\end{equation*}
$$

Next session, when they reconsider the original problem, Mr. Ford introduces the class to the envelope curve (figure 11). Some experimentation convinces the class that this curve can be used to determine whether the ladder fits.


Figure 11: Envelope curve of the ladder, along with its equation
The symbolic geometry system generates the equation of the envelope and the class is quick to copy this equation into their algebra system and solve for L . They get six rather long and complicated solutions. Mr. Ford points out, however, that one solution is positive real, the other negative real, and the others involve the symbol i , which, he reminds them, is the square root of negative one and not appropriate for a length.

$$
\frac{\sqrt{\left(y^{2} x\right)^{(1 / 3)} \nsupseteq \dot{\mathrm{e}} y^{2}\left(y^{2} x\right)^{(1 / 3)}+3 x\left(y^{2} x\right)^{(2 / 3)}+3 y^{2} x+x^{2}\left(y^{2} x\right)^{(1 / 3)} \ddot{\partial}}}{\left(y^{2} x\right)^{(1 / 3)}}
$$

Mr. Ford suggests plugging this value back in for L and they verify that with this length the corner lies on the envelope curve and this is the critical length of the ladder.

He asks the class if they would expect the result to be symmetric in $x$ and $y$. The consensus is, yes it should be, as the maximum length of ladder which can get round a corner from a corridor of width x into a corridor of width y is the same as the maximum length of a ladder which can go in the other direction. Mr. Ford points out that the expression as written is not obviously symmetric in x and y - for example, the denominator contains x and y to different powers. He challenges them to simplify the expression so that its symmetry is clear.

The first attempt of the class is to apply simplify() in their CAS, but it returns the same result as was fed in. After a flurry of pencil and paper work, a symmetric answer is produced:

$$
\begin{equation*}
\sqrt{y^{2}+3 x^{\frac{4}{3}} y^{\frac{2}{3}}+3 y^{\frac{4}{3}} x^{\frac{2}{3}}+x^{2}} \tag{10}
\end{equation*}
$$

Meanwhile one student, lazier than the rest, but with a better knowledge of his algebra system has persuaded it to simplify the expression by applying assumptions that x and y are positive real.
$>$ simplify (\%) assuming $x>0, y>0$;

$$
\left(y^{(2 / 3)}+x^{(2 / 3)}\right)^{(3 / 2)}
$$

You know what we have to do now, Mr. Ford says, to groans from the class - reconcile the solutions. All but one class member is allowed to use their algebra system. He points out that the power $3 / 2$ can be regarded as the square root of a cube, so long as everything is positive:
$>\operatorname{expand}\left(\left(x^{\wedge}(2 / 3)+y^{\wedge}(2 / 3)\right)^{\wedge} 3\right) ;$

$$
x^{2}+3 x^{(4 / 3)} y^{(2 / 3)}+3 x^{(2 / 3)} y^{(4 / 3)}+y^{2}
$$



Figure 12: Solution to the longest ladder problem
Mr. Ford looks at the clock and sees there are 5 minutes left in class. Verification, he says - can anyone give me a known case we can use to verify this result? They set $y=x$ in the formula and observe that it simplifies to (11).

$$
\begin{equation*}
\left(2 x^{\frac{2}{3}}\right)^{\frac{3}{2}}=2^{\frac{3}{2}} x=2 \sqrt{2} x \tag{11}
\end{equation*}
$$

The class files out, and we cross the hall where Ms. Franklin is drawing a calculus problem on the board. The sport of rugby, she says, is a lot like football here in America, but when you score a touchdown (they call it a "try") you take the extra point kick (they call it a "conversion") from a place in the field in line with the place the try was scored. So if the try is scored close to the sideline, the kick is taken from somewhere close to the sideline. If it is scored in the middle of the field, the kick can be taken from the middle of the field (fig. 13).


Figure 13: Rugby extra point kick is taken from a position on the field in line with the location of the touchdown. $G$ is the width of the goal, and d the distance from the nearest post to the location of the touchdown.

My question is: if the goalposts are width g , and the try is scored distance d outside the left post, what is the best place to take the kick. She asks the class to set up the problem in their geometry system.

The class now discusses how to phrase the question as an optimization problem: what is to be optimized. An initial suggestion that they should minimize the length of the kick is disposed of with the realization that the minimum distance would be found on the goal line itself, which would lead to a need to bend the kick to stand any chance of scoring.

Ms. Franklin suggests maximizing the angle made by the goalposts at the point of kick, and the class quickly derives the angle from their geometry system (figure 14)


Figure 14: Angle made by the goalposts at the point of the kick
The class agrees that assuming the kicker is able to make the distance, the best place to kick will be where this angle is biggest.

We are going to be doing this problem by hand, says Ms. Franklin, but first, let's say x and y are angles between 0 and 90 , and I tell you that y is greater than x , what can you tell me about $\tan (\mathrm{y})$ and $\tan (\mathrm{x})$ ? Think about the graph of the tangent function - that's right, $\tan (\mathrm{y})>\tan (\mathrm{x})$. So if I tell you I've found the biggest possible value for the angle in our problem, then that will also be the biggest possible value of $\tan ()$ ? The class agrees, and is prepared to concede that a maximum angle will occur where the argument of the arctan is maximized. An almost audible relief settles over the class as they realize they don't need to differentiate the $\arctan$.

In due course, an answer is reached for the derivative of the argument

$$
\begin{equation*}
\frac{g x^{2}-g d^{2}-g^{2} d}{\left(d^{2}+d g+x^{2}\right)^{2}} \tag{12}
\end{equation*}
$$

(12) is solved for x to yield:

$$
\begin{equation*}
x=\sqrt{d^{2}+d g} \tag{13}
\end{equation*}
$$

Feed (13) back into your geometry diagram, suggests Ms. Franklin, and can we see the locus of the best kick locations for different values of d? Ms. Franklin asks what kind of curve this looks like. Parabola and hyperbola are put forward as ideas. The weight of popular opinion is behind hyperbola. Can we work out the equation of this curve, Ms. Franklin asks. We haven't established a coordinate location for our diagram yet, I suggest we make the origin the center of the goalposts:


Figure 15: Locus of optimal kick points is a hyperbola whose asymptote is the line $Y=X$
Popular opinion is vindicated by the curve's equation (fig. 15). Hyperbolas have asymptotes, says Ms. Franklin, what is the asymptote of this one? When X and Y are large, she prompts, will the size of g matter? So if we ignore g , what is this equation? The class deduces the asymptote is the line $\mathrm{Y}=\mathrm{X}$ and place it on their diagram.

So what practical advice, would this class give a rugby kicker? After discussion on how to measure angles on a rugby field, they come up with this statement,
"Kick from approximately the same distance out from the goal line as the try is from the center of the field."

As the class dissolves we return to our own decade to try and draw conclusions from our excursion into the future.

## Conclusion

Reviewing the fictional classroom experiences described above, we try to identify the precise points of contact between the algebraic and the geometric components of the problems.

In the incircle / excircle example, the algebraic expression for the radius of the incircle is observed to be closely related to the expression for the area of the triangle. This observation directs the search for a geometric proof in terms of a decomposition of areas. In its turn, the geometry provides context for algebraic manipulation, where the expression for the area in terms of the incircle radius needs to be inverted to give the radius in terms of the area. Further, the geometry provides an opportunity to pose an algebraic discovery problem. Strikingly, in this example the students are asked to discover geometric theorems by manipulating algebraic expressions.

Algebraic symmetry is a further point of contact. The expression for the radius of the incircle is symmetric in all three lengths. This correlates to the fact that the incircle is defined symmetrically with respect to each side of the triangle. An excircle on the other hand is defined symmetrically with respect to two of the sides but not with respect to the third. The corresponding algebraic expression is symmetric with respect to two of the lengths but not the third.

Symmetry also plays a role in the second example. It is geometrically obvious that the longest ladder which can turn a corner from a corridor of width x into a corridor of width y is the same as the longest ladder which can turn a corner in the opposite direction. The initial form of the algebraic result, however, is not symmetric in x and y and this provides the motivation for an algebraic simplification assignment. Algebraic simplification can seem an academic exercise and somewhat arbitrary, but in this case, the goal of exposing the inherent symmetry of the expression gives a strong motivation and sense of direction to the manipulation.

An elegant mathematical result for a real problem is achieved through a combination of symbolic geometry, computer algebra and old-fashioned by-hand algebraic manipulation. The symbolic geometry is essential to compute the equation of the envelope to the ladder as it moves round the corner. Although the definition of the envelope curve can be grasped intuitively, the mathematics behind computing its equation is beyond students at this level. Likewise, solving a non factoring cubic is something best left to a CAS.

However, the student is not a passive participant in the process. In order to drive the geometrical end of the problem, he needs to have a good understanding of loci, and of parametric and implicit equations of curves. In order to interpret the results from the CAS, he needs to be familiar with complex numbers, and he needs the mathematical common sense to eliminate geometrically meaningless results. Finally, he needs good algebraic manipulation skills of his own in order to massage the automatic results into a desired form, or to identify that different forms of an algebraic expression are identical.

In the third example, geometry is initially used in a traditional role in a calculus class: providing the context for an optimization problem. As the symbolic geometry software is able to simply derive the expression for the angle, the student is able to focus on the task at hand: using calculus to solve an optimization problem. Here, the role of the technology is in isolating the component of a problem which is relevant to the current lesson. The algebraic results of the calculus problem are fed back into the geometry software in the form of a curve. The algebraic equation of the curve indicates its form and suggests the use of an asymptote as a linear approximation. The asymptote in turn can be computed from the curve equation. At this point the distinction between algebraic and geometric representation has become blurred; arguably a desirable conclusion.

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